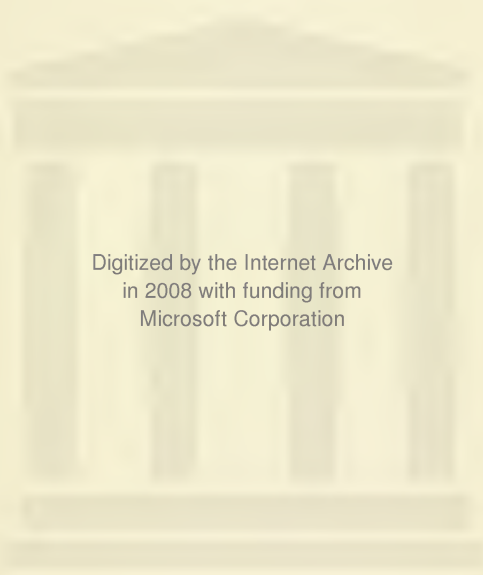




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MESSENGER OF MATHEMATICS.

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CONTENTS TO VOL. XXXVIII.

	PAGE
Notes on some points in the integral calculus. By G. H. HARDY - -	1
Notes on integral equations. By H. BATEMAN - - - -	8
On the last theorem of Fermat. By L. E. DICKSON - - - -	14
On surfaces which have assigned families of curves as their lines of curvature. By A. R. FORSYTH - - - - -	33
A note on trigonometrical series. By W. H. YOUNG - - - -	44
A formula in finite differences and its application to mechanical quadrature. By S. T. SHOVELTON - - - - -	49
Generalization of the differential operative symbol with an extended form of Boole's equation $\theta(\theta-1)(\theta-2)\dots(\theta-n+1) = x^n \frac{d^n}{dx^n}$. By F. H. JACKSON - - - - -	57
q -form of Taylor's theorem. By F. H. JACKSON - - - -	62
On the derivates of non-differentiable functions. By W. H. YOUNG -	65
Notes on integral equations. By H. BATEMAN - -	70
The integral of the product of three zonal harmonics. By D. EDWARDES	76
Diophantine, &c., factorisation of quartans. By LT.-COL. A. CUNNINGHAM - - - - -	81, 145
A theorem of Cayley's regarding Sturm's functions. By T. MUIR - -	105
On the harmonic conic of two given conics. By E. G. HOGG - - -	111
List of tables of $\int_0^x e^{-t^2} dt$ and connected integrals and functions. By J. W. L. GLAISHER - - - - -	117
On certain definite integrals whose values can be expressed in terms of Bessel's functions. By G. H. HARDY - - - - -	129
On the order of the group of automorphisms of an Abelian group. By H. HILTON - - - - -	132
On a class of regions. By J. MERCER - - - - -	135
The summation of certain series. By J. H. C. SEARLE - - - -	138
Notes on some points in the integral calculus. By G. H. HARDY - -	176
The Jacobian of four quadrics. By C. M. JESSOP - - - -	185
A formula in interpolation. By C. S. JACKSON - - - - -	187

ERRATUM.—p. 65, at end of (3) add “, *exception being made of at most another set of the first category.*”

MESSENGER OF MATHEMATICS.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XXIII.

On certain oscillating cases of Dirichlet's Integral.

§ 1. It is well known—the result is indeed the first foundation of the theory of Fourier's Series—that if $f(x)$ has a limited total fluctuation in the interval $(0, \xi)$, and $f(x) \rightarrow f(+0)$ as $x \rightarrow 0$, then the integral

$$(1) \quad J(\lambda) = \int_0^\xi \frac{\sin \lambda x}{x} f(x) dx$$

tends to the limit $\frac{1}{2}\pi f(+0)$ as $\xi \rightarrow \infty$.

The integral (1) is commonly described as 'Dirichlet's Integral.' It is known that continuous functions exist for which this result does not hold; examples of such functions have been given by Du Bois-Reymond and Schwarz.*

The former of these two writers also considered another similar problem. Suppose that $f(x)$ is continuous and of limited total fluctuation in (δ, ξ) , for any positive value of δ , but has an oscillatory discontinuity for $x=0$; the simplest example of such a function is

$$f(x) = F(x) \phi(1/x) \frac{\cos}{\sin} \psi(1/x),$$

* See Hobson, *Theory of Functions of a Real Variable*, pp. 701 *et seq.*

where $F(x)$ is regular in every respect near $x=0$, and $\phi(1/x)$ and $\psi(1/x)$ are monotonic functions of which the second tends to ∞ as $x \rightarrow 0$, and which it is natural to select from among the members of the standard logarithmic scale. Thus

$$(1/x^\mu) \sin(1/x^\nu), \quad \sin\{\log \log(1/x)\}$$

would be appropriate forms of $f(x)$. Then what are the conditions which $\phi(1/x)$ and $\psi(1/x)$ must satisfy in order that Dirichlet's integral may tend to a limit or oscillate, finitely or infinitely, as $\lambda \rightarrow \infty$? These conditions will naturally take the form of hypotheses as to the respective rates of increase of $\phi(1/x)$ and $\psi(1/x)$.

Du Bois-Reymond considered this question in great detail,* and proved a large number of very interesting results. His analysis, however, is unfortunately very elaborate and difficult to follow, and his use of the ideas and notation of his favourite '*Infinitärrechnung*' is rather bewildering to those who are better accustomed to reasoning which proceeds by means of inequalities.

In this and some succeeding notes I propose to restate (sometimes in a more general or precise form) some of Du Bois-Reymond's simplest and most striking results, to give simpler proofs of some of them, and to illustrate them by means of particular integrals which are interesting in themselves and can be evaluated either exactly or asymptotically. Some of the more difficult results can only be proved by more elaborate methods.

§ 2. The simplest case is that in which the oscillations of $f(x)$, as $x \rightarrow 0$, are comparatively slow. Du Bois-Reymond showed that the case in which

$$\phi(1/x) = \log(1/x)$$

is a *critical* case, and that the answers to the question are entirely different according as $\psi(1/x) \rightarrow \infty$ less or more rapidly than $\log(1/x)$.

§ 3. Let us suppose that $f(x)$ has a continuous derivative for $x > 0$, and that $xf(x) \rightarrow 0$ with x . Also let

$$\{f(x)\}/x = \chi(x) = \chi(1/u) = \theta(u),$$

* See his memoir in the *Abhandlungen der bayerischen Akademie*, bd. xii., Abthg. 2.

so that $\theta(u) = uf(1/u)$. Then

$$\begin{aligned}\int_0^\xi \frac{\sin \lambda x}{x} f(x) dx &= \int_0^\xi \theta(u) \frac{d}{dx} \left(\frac{1 - \cos \lambda x}{\lambda} \right) dx \\ &= \theta(1/\xi) \frac{1 - \cos \lambda \xi}{\lambda} - \int_0^\xi \frac{1 - \cos \lambda x}{x} \frac{d}{dx} \theta(u) dx,\end{aligned}$$

since $\xi^2 \theta(1/\xi) = \xi f(\xi) \rightarrow 0$ as $\xi \rightarrow 0$. The first term obviously tends to zero as $\lambda \rightarrow \infty$, and may therefore be ignored. The second may be written in the form

$$(2) \quad I(\lambda) = \int_0^\xi \theta'(u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx.$$

Now let $\mu(\lambda)$ be a function of λ which tends to ∞ with λ , but more slowly than λ , so that $\mu \rightarrow \infty$ and $\mu/\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$, and let

$$\delta_1 = 2/\lambda\mu, \quad \delta_2 = 2\mu/\lambda.$$

Also let

$$\begin{aligned}(3) \quad I(\lambda) &= \left(\int_0^{\delta_1} + \int_{\delta_1}^{\delta_2} + \int_{\delta_2}^\xi \right) \theta'(u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx \\ &= I_1(\lambda) + I_2(\lambda) + I_3(\lambda).\end{aligned}$$

§ 4. Now let us introduce a fresh condition, viz. that

$$|\theta'(u)| < K,$$

where K is a constant, for $u \geq 1/\xi$. Then

$$|I_1(\lambda)| < K \int_0^{\delta_1} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx < K \int_0^{1/\mu} \left(\frac{\sin t}{t} \right)^2 dt,$$

$$|I_3(\lambda)| < K \int_{\delta_2}^\xi \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx < K \int_\mu^\infty \left(\frac{\sin t}{t} \right)^2 dt,$$

and so

$$I_1(\lambda) \rightarrow 0, \quad I_3(\lambda) \rightarrow 0,$$

as $\lambda \rightarrow \infty$. Hence

$$\begin{aligned}I(\lambda) &= I_2(\lambda) + \varepsilon_\lambda = \theta'(\rho) \int_{\delta_1}^{\delta_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx + \varepsilon_\lambda \\ &= \theta'(\rho) \int_{1/\mu}^\mu \left(\frac{\sin t}{t} \right)^2 dt + \varepsilon_\lambda \\ (4) \quad &= \theta'(\rho) \Omega(\lambda) + \varepsilon_\lambda,\end{aligned}$$

where $\Omega(\lambda) \rightarrow \frac{1}{2}\pi, \quad \varepsilon_\lambda \rightarrow 0,$ *

and $\lambda/\mu < \rho < \lambda\mu.$

§ 5. Now, if

$$f(x) = \phi(1/x) \sin \psi(1/x),$$

we have

$$\theta(u) = uf(1/u) = u\phi(u) \sin \psi(u),$$

$$\theta'(u) = \phi(u) \sin \psi(u) + u\phi'(u) \sin \psi(u) + u\phi(u) \psi'(u) \cos \psi(u).$$

Let us suppose first that

$$\phi(1/x) \equiv 1,$$

so that $\theta'(u) = \sin \psi(u) + u\psi'(u) \cos \psi(u).$

Further, let us suppose that ψ is a member of the logarithmic scale lower than $\log(1/x)$, such as

$$\{\log(1/x)\}^\alpha \{\log \log(1/x)\}^\beta \quad (0 < \alpha < 1).$$

Then $\psi'(u)$ is ultimately monotonic, and

$$u\psi'(u) \rightarrow 0,$$

as $u \rightarrow \infty$.† Hence

$$\theta'(\rho) = \sin \psi(\rho) + \varepsilon_\lambda,$$

where $\varepsilon_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$.

§ 6. It follows that

$$\begin{aligned} (5) \quad I(\lambda) &= \{\sin \psi(\rho) + \varepsilon_\lambda\} \left(\frac{1}{2}\pi + \varepsilon_\lambda\right) + \varepsilon_\lambda \\ &= \frac{1}{2}\pi \sin \psi(\rho) + \varepsilon_\lambda, \end{aligned}$$

* In what follows I use ε_λ for 'a function of λ whose limit of $\lambda \rightarrow \infty$ is 0.' It is, of course, not the *same* function in different inequalities.

† All members of the logarithmic scale are ultimately monotonic. Their derivatives are also ultimately monotonic, and the derivative of any member has an increase the same as that of some other member. Thus the derivative of

$$u^\alpha (\log u)^\beta (\log_2 u)^\gamma \dots (\log_k u)^\kappa$$

behaves like

$$u^{\alpha-1} (\log u)^\beta (\log_2 u)^\gamma \dots (\log_k u)^\kappa.$$

And the order of the derivatives in the scale is the same as that of the original functions. Thus, if $\psi(u)$ is in the scale and $\psi(u)/\log u \rightarrow 0$, so also $u\psi'(u) \rightarrow 0$.

where $\lambda/\mu < \rho < \lambda\mu$, provided only $\mu \rightarrow \infty$ with λ , no matter how slowly. But it is easy to see also that

$$\psi(\rho) = \psi(\lambda) + \varepsilon_\lambda.$$

For
$$\psi'(u) = 1/\{u\chi(u)\},$$

where χ is a function which is ultimately monotonic and which tends to ∞ with u ; and

$$|\psi(\rho) - \psi(\lambda)| = \left| \int_\lambda^\rho \frac{du}{u\chi(u)} \right| < \frac{\log \mu}{\chi(\lambda/\mu)},$$

and, no matter however slowly χ may tend to ∞ , we can always find a function μ which tends to ∞ so slowly that

$$\frac{\log \mu}{\chi(\lambda/\mu)} \rightarrow 0.$$

Hence, we have finally

$$(6) \quad I(\lambda) = \frac{1}{2}\pi \sin \psi(\lambda) + \varepsilon_\lambda,$$

so that $I(\lambda)$ oscillates as $\lambda \rightarrow \infty$.

§7. Let us now consider the case in which $\phi(1/x)$ is a member of the logarithmic scale other than unity, or, more generally, in which

$$f(x) = F(x) \phi(1/x) \sin \psi(1/x),$$

where $\phi(1/x)$ is a member of the scale and $F(x)$ is regular in every way at the origin. Then

$$\theta(u) = uF(1/u) \phi(u) \sin \psi(u),$$

and, provided $\phi(u)$ is lower in the scale than $\log u$ (so that $u\phi'(u) \rightarrow 0$), the dominant term of $\theta(u)$ is

$$F(1/u) \phi(u) \sin \psi(u).$$

If $\phi(u) \equiv 1$, we obtain, by analysis practically identical with that of the preceding sections,

$$(7) \quad I(\lambda) = \frac{1}{2}\pi F(0) \sin \psi(\lambda) + \varepsilon_\lambda.$$

And if $\phi(u) \rightarrow 0$ as $u \rightarrow \infty$, we obtain

$$(8) \quad I(\lambda) \rightarrow 0.$$

But if $\phi(u) \rightarrow \infty$ with u our analysis fails and considerably more elaborate methods are needed to show that (if $\phi(u)$

is subject to certain restrictions) $I(\lambda)$ oscillates infinitely as $\lambda \rightarrow \infty$. But the preceding argument is sufficient to find the limits of convergence and oscillation of $J(\lambda)$, as defined by means of the increase or decrease of ϕ . The general result may be summed up thus: *if $\phi(1/x)$ and $\psi(1/x)$ are members of the logarithmic scale, and the increase of $\psi(1/x)$ is less than that of $\log(1/x)$, and $F(x)$ is regular for $x=0$, then*

$$J(\lambda) = \int_0^{\epsilon} F(x) \phi(1/x) \sin \psi(1/x) \frac{\sin \lambda x}{x} dx$$

tends to the limit zero, as $\lambda \rightarrow \infty$, if $\phi(u) \rightarrow 0$ as $u \rightarrow \infty$, oscillates finitely, like

$$\frac{1}{2} \pi F(0) \sin \psi(\lambda),$$

if $\phi(u) \equiv 1$, and oscillates infinitely (though this we have not proved) if $\phi(u) \rightarrow \infty$.

Of course $\sin \psi(1/x)$ may be replaced by $\cos \psi(1/x)$ in all that precedes.

§ 8. I shall now consider the case in which

$$\psi(1/x) = m \log(1/x), \quad (m > 0).$$

In this case the preceding argument is invalid owing to the facts (1) that the approximate formula for $\theta'(\rho)$ given in § 5 no longer holds and (2) that we can no longer apply the argument of § 6. We therefore proceed differently.

Suppose that

$$f(x) = F(x) \phi(1/x) \sin \{m \log(1/x)\},$$

where $\phi \equiv 1$ or $\phi \rightarrow 0$ as $x \rightarrow 0$, $F(x)$ is regular for $x=0$, and $F(0)=0$. As before, we have $|\theta'(u)| < K$ and the equation (4) of § 4 still holds.

Also $\theta'(u) \rightarrow 0$ as $u \rightarrow \infty$, and, if Θ is the maximum of $|\theta'(u)|$ in the interval $(\lambda/\mu, \lambda\mu)$, we have

$$|I_2(\lambda)| < \frac{1}{2} \pi \Theta \rightarrow 0$$

as $\lambda \rightarrow 0$. Hence $J(\lambda) \rightarrow 0$.

It follows that, if $F(0) \neq 0$, the behaviour of $J(\lambda)$ as $\lambda \rightarrow \infty$, at any rate when $\phi \equiv 1$ or $\phi \rightarrow 0$, is the same as that of

$$F(0) \int_0^{\epsilon} \phi(1/x) e^{-\rho x} \sin \{m \log(1/x)\} \frac{\sin \lambda x}{x} dx,$$

where ρ is any positive number. If $\phi(1/x) \rightarrow 0$ there is no difficulty in seeing that $J(\lambda) \rightarrow 0$: the interesting case is that in which $\phi \equiv 1$.

§ 9. We therefore consider the integral

$$\int_0^{\xi} e^{-\rho x} \sin \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx;$$

and we can evidently replace the upper limit by ∞ . For, given ε , we can choose ξ_1 so that

$$\left| \int_{\xi_1}^{\infty} \right| < \varepsilon$$

for all values of λ , and then, by Dirichlet's theorem, we can choose λ so that

$$\left| \int_{\xi}^{\xi_1} \right| < \varepsilon.$$

But it is easy to prove the formulæ

$$\begin{aligned} \int_0^{\infty} e^{-\rho x} \cos \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx &= [\alpha \sin \{\tfrac{1}{2}m \log (\lambda^2 + \rho^2)\} \\ &\quad - \beta \cos \{\tfrac{1}{2}m \log (\lambda^2 + \rho^2)\}] \sinh \{m \arctan (\lambda/\rho)\}, \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} e^{-\rho x} \sin \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx &= -[\alpha \cos \{\tfrac{1}{2}m \log (\lambda^2 + \rho^2)\} \\ &\quad + \beta \sin \{\tfrac{1}{2}m \log (\lambda^2 + \rho^2)\}] \sinh \{m \arctan (\lambda/\rho)\}, \end{aligned}$$

which hold for $\lambda > 0$, $\rho \geq 0$, and assume, for $\rho = 0$, the forms

$$\begin{aligned} \int_0^{\infty} \cos \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx \\ = \{\alpha \sin (m \log \lambda) - \beta \cos (m \log \lambda)\} \sinh \tfrac{1}{2}m\pi, \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} \sin \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx \\ = -\{\alpha \cos (m \log \lambda) + \beta \sin (m \log \lambda)\} \sinh \tfrac{1}{2}m\pi. \end{aligned}$$

Here α and β are defined by the equation

$$\Gamma(mi) = \alpha + \beta i$$

$$\text{and } \alpha^2 + \beta^2 = \Gamma(mi) \Gamma(-mi)$$

$$= \{\Gamma(mi) \Gamma(1-mi)\} / (-mi) = \pi / (m \sinh m\pi).$$

Hence we have the asymptotic formulæ

$$\begin{aligned} \int_0^\infty e^{-\rho x} \cos \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx \\ = \sinh \frac{1}{2} m \pi \sqrt{\{(m/\pi) \sinh m \pi\}} \sin (m \log \lambda - \theta), \\ \int_0^\infty e^{-\rho x} \sin \{m \log (1/x)\} \frac{\sin \lambda x}{x} dx \\ = - \sinh \frac{1}{2} m \pi \sqrt{\{(m/\pi) \sinh m \pi\}} \cos (m \log \lambda - \theta), \end{aligned}$$

where $\theta = \arctan (\beta/\alpha)$.

And so the asymptotic formula for $J(\lambda)$ is in this case

$$- F(0) \sinh \frac{1}{2} m \pi \sqrt{\{(m/\pi) \sinh m \pi\}} \cos \log (\lambda - \theta),$$

and the result of § 7 must be modified in this way when $\psi(1/x) = m \log (1/x)$.

NOTES ON INTEGRAL EQUATIONS.

I.

By *H. Bateman*.

The integral equation of the first kind.

SOME progress can be made in the theory of the integral equation of the first kind

$$f(s) = \int_a^b \kappa(s, t) \phi(t) dt \dots\dots\dots(1),$$

in which $\kappa(s, t)$ is a real continuous symmetric function of s and t within the region defined by the inequalities $a \leq s \leq b$, $a \leq t \leq b$, by splitting the function $\kappa(s, t)$ into two parts and introducing a parameter λ .

$$\text{Let} \quad \kappa(s, t) = f(s, t) - \lambda g(s, t) \dots\dots\dots(2),$$

where $f(s, t)$ is a definite function*, i.e. a real symmetric

* Some properties of definite functions are considered in a paper by the author, *Messenger of Mathematics*, vol. XXVII., pp. 91-95.

function such that the double integral

$$\int_a^b \int_a^b f(s, t) x(s) x(t) ds dt$$

is positive for every real continuous function $x(s)$.

Equation (1) may now be written

$$f(s) = \int_a^b [f(s, t) - \lambda g(s, t)] \phi(t) dt \dots \dots \dots (3).$$

We shall suppose that the function $f(s, t)$ is chosen so that the integral equation

$$f(s) = \int_a^b f(s, t) g(t) dt \dots \dots \dots (4)$$

can be solved for $g(t)$ and that a set of conditions (A) are known which are such that, when $f(s)$ satisfies these conditions, the function $g(t)$ is continuous for the range ($a \leq t \leq b$).

There is only one continuous function $g(t)$ which satisfies equation (4) under these conditions, because the homogeneous equation*

$$0 = \int_a^b f(s, t) \chi(t) dt \dots \dots \dots (5)$$

cannot be satisfied by a continuous function $\chi(t)$ if $f(s, t)$ is definite.

Supposing the function $f(s)$ to satisfy the conditions (A) we may endeavour to satisfy (3) by assuming

$$\phi(s) = g(s) + \lambda \int_a^b K(s, t) g(t) dt \dots \dots \dots (6),$$

where $K(s, t)$ is a function depending on λ which is independent of $f(s)$ and $g(s)$.

When this value of $\phi(s)$ is substituted in equation (3) the resulting equation will be satisfied for every continuous function $g(s)$, if

$$g(s, x) = \int_a^b [f(s, t) - \lambda g(s, t)] K(t, x) dt \dots (7).$$

The function $K(s, x)$ when it exists will be called the *solving-function* of the integral equation (3), it is evidently a solution of (3) for the case in which

$$f(s) = g(s, x).$$

* *Op. cit.*, p. 91.

Now in order that equation (7) may possess a continuous solution for $\lambda = 0$ it seems necessary to assume that $g(s, x)$ satisfies the conditions (A). Under these circumstances we are able to solve the equation

$$g(s, x) = \int_a^b f(s, t) \kappa(t, x) dt \dots \dots \dots (8),$$

and obtain a continuous function $\kappa(t, x)$.

If the values of $f(s)$ and $g(s, x)$ given by (4) and (8) are substituted in (3) and (7) they become

$$\int_a^b f(s, t) \left[g(t) - \phi(t) + \lambda \int_a^b \kappa(t, x) \phi(x) dx \right] dt = 0$$

and

$$\int_a^b f(s, t) \left[\kappa(t, x) - K(t, x) + \lambda \int_a^b K(y, x) \kappa(t, y) dy \right] dt = 0$$

respectively; accordingly, since $f(s, t)$ is a *definite function* and equation (5) is impossible, we must have

$$g(t) = \phi(t) - \lambda \int_a^b \kappa(t, x) \phi(x) dx \dots \dots \dots (9)$$

and

$$\kappa(t, x) = K(t, x) - \lambda \int_a^b \kappa(t, y) K(y, x) dy \dots (10).$$

These equations show that $K(t, x)$ is the solving function of the integral equation (9) which is an equation of the second kind.

Now the theory of the integral equation of the second kind tells us that $K(t, x)$ can be expressed in the form*

$$K(t, x) = \frac{D(\lambda; t, x)}{D(\lambda)} \dots \dots \dots (11),$$

where $D(\lambda; t, x)$ and $D(\lambda)$ are integral functions of λ , and that $\phi(t)$ is given uniquely by the formula (6) except for the *singular values* of λ which make $D(\lambda) = 0$.

If λ is singular the homogeneous equation

$$0 = \psi(s) - \lambda \int_a^b \kappa(s, t) \psi(t) dt \dots \dots \dots (12)$$

* I. Fredholm, *Acta Mathematica* (1903).

possesses a solution, and then the corresponding homogeneous equation

$$0 = \int_a^b [f(s, t) - \lambda g(s, t)] \psi(t) dt \dots \dots \dots (13)$$

is also satisfied.

Conversely, if a solution of (13) exists, we may deduce that (12) is also satisfied and that λ is a root of the transcendental equation

$$D(\lambda) = 0.$$

Hence the singular values of λ , for which the homogeneous equation (13) can be satisfied, are the zeros of an integral function $D(\lambda)$, provided $g(s, t)$ is a symmetric function satisfying the conditions (A).

This last restriction is introduced because it appears as if we might in some cases write the equation in the form

$$0 = \int_a^b [g(s, t) - (1/\lambda)f(s, t)] \psi(t) dt$$

and deduce in a similar way that the possible values of $1/\lambda$ were also zeros of an integral function $E(\mu)$. This, however, would imply either that the aggregate of singular values of λ had a point of condensation at $\lambda=0$, or that it was finite. The first requirement is incompatible with the result that the singular values of λ are the zeros of an integral function. The second is admissible, but only when equation (13) is of a particular type.

We have shown elsewhere* that if $f(s, t)$ is a *definite function* and $g(s, t)$ is a real continuous symmetric function the values of λ for which equation (13) can be satisfied are all real. We shall now show that they are at most *simple* poles of the function $\phi(t)$ which satisfies equation (3).

To prove this we suppose that λ_0 is a singular value of λ and that $\phi(s)$ can be expanded in the vicinity of $\lambda=\lambda_0$ in the form

$$\phi(s) = \frac{\phi_n(s)}{(\lambda_0 - \lambda)^n} + \frac{\phi_{n-1}(s)}{(\lambda_0 - \lambda)^{n-1}} + \dots + \frac{\phi_1(s)}{\lambda_0 - \lambda} + \phi_0(s) + F(s) \dots (14),$$

where $F(s)$ is regular in the vicinity of $\lambda=\lambda_0$, vanishing for $\lambda=\lambda_0$, and the functions $\phi_r(s)$ are independent of λ .

* *Camb. Phil. Trans.*, vol. xx, (1908), p. 371.

Substituting in (3) and equating to zero the coefficients of the negative powers of $\lambda_0 - \lambda$, we obtain the system of equations

$$0 = \int_a^b [f(s, t) - \lambda_0 g(s, t)] \phi_n(t) dt \dots (15),$$

$$\int_a^b g(s, t) \phi_n(t) dt = \int_a^b [f(s, t) - \lambda_0 g(s, t)] \phi_{n-1}(t) dt \dots (16),$$

.....

$$f(s) + \int_a^b g(s, t) \phi_1(t) dt = \int_a^b [f(s, t) - \lambda_0 g(s, t)] \phi_0(t) dt \dots (17).$$

A solution of (15) exists since λ_0 is a singular value of λ . If, however, $n > 1$ equation (16) cannot be satisfied, for if we multiply by $\phi_n(s)$ and integrate, the right-hand side disappears on account of (15), and we get

$$\int_a^b \int_a^b g(s, t) \phi_n(s) \phi_n(t) ds dt = 0.$$

But we also have

$$\int_a^b \int_a^b [f(s, t) - \lambda_0 g(s, t)] \phi_n(s) \phi_n(t) ds dt = 0,$$

therefore
$$\int_a^b \int_a^b f(s, t) \phi_n(s) \phi_n(t) ds dt = 0.$$

This, however, is impossible if $\phi_n(s)$ is not identically zero, since $f(s, t)$ is a *definite* function, hence n cannot be greater than one. The case of $n = 1$ is different, because then equation (16) is replaced by (17) and the above argument no longer holds.

In particular, by taking $f(s) = g(s, x)$ we conclude that the singular values of λ are simple poles of the function $K(s, x)$, provided, of course, that $g(s, x)$ satisfies conditions (A).

If $g(s, x)$ does not satisfy conditions (A) we are not justified in assuming that the values of λ are poles of $K(s, x)$, but when conditions (A) are satisfied and (7) is replaced by (10) we may deduce the fact from the properties of the integral equation of the second kind.

The theorem we have just proved is of some importance in the theory of the integral equation of the second kind, because it tells us that, if the characteristic function $\kappa(s, t)$ is such that a definite function $f(s, t)$ can be found which will make the integral

$$\int_a^b f(s, x) \kappa(x, t) dx$$

a *symmetric function* of s and t , then the singular values of λ are all *real* and are only *simple poles* of the solving function $K(s, t)$.

The theory of the integral equation (9) is known to be much simplified when the poles of $K(s, t)$ are all simple,* and the above theorem often enables us to detect when this is the case.

For instance, in the problems occurring in the potential theory, the integral equations have as their characteristic function the function†

$$\kappa(p, q) = \frac{d\Omega}{dS_p} = \frac{\cos \phi}{r_{pq}^2},$$

where $d\Omega$ is the solid angle subtended at a point Q on a closed surface S by a surface element dS at another point P on the surface, r_{pq} is the distance, PQ and ϕ is the angle between the normal at P and the radius vector PQ .

Now the function $1/r_{pq}$ is a *definite function*, for the double integral

$$\iint \frac{f(p)f(q)}{r_{pq}} dS_p dS_q$$

represents twice the energy of a certain surface distribution, and so is necessarily positive. Further, Plemelj‡ has shown that the function

$$\int \frac{dS_p}{r_{pq'}} \cdot \frac{\cos \phi}{r_{pq}} = g(q, q')$$

is a *symmetric function* of the parameters of the point Q and Q' , accordingly, if we extend our theorem to the case in which there are several variables and take into account the fact that the results still remain valid when the functions become infinite in a certain way, we may obtain the known theorem§ that the singular values of λ for the integral equation with the characteristic function $\kappa(p, q)$ are all *real* and are *simple poles* of the corresponding solving function $K(p, q)$.

* J. Plemelj, *Monatshefte für Mathematik und Physik* (1904), pp. 125–128.

† The symbol p is used to denote that κ is a function of the two parameters which fix the position of the point P on the surface S .

‡ *Monatshefte für Mathematik und Physik* (1904), pp. 341–346.

§ *Ibid.*, pp. 380–383.

ON THE LAST THEOREM OF FERMAT.

By Professor L. E. Dickson.

1. FERMAT stated (but did not publish a proof) that

$$(1) \quad u^n + v^n + w^n = 0 \quad (n \text{ a positive integer } > 2)$$

is not possible in integers u, v, w , each not zero. No correct general proof has yet been given, although many writers* have attacked the problem. Doubtless the most important work in this direction is that by Kummer, whose proof applies to an extensive class of prime numbers n , viz. those for which certain Bernoullian numbers are not divisible by n . But it is quite laborious to test whether or not a given prime n falls in Kummer's class.

All writers on the subject have found it necessary to separate the case in which no one of the integers u, v, w is divisible by the prime n from that in which at least one is divisible by n . For the first case a very simple method (see §§ 2-3) was invented by Sophie Germain, and generalized by Legendre,† to apply to every prime n for which one of the numbers $mn + 1$, $m = 2, 4, 8, 10, 14, 16$ is a prime (the exceptional character of $n = 3$ for $m = 10$ and 14 being overlooked). In the present paper I obtain this result by a direct general analysis, and extend it to the new cases $m = 20, 22, 26, 28, 32, 40, 56, 64$ (the exceptional values of n being given in § 23).

The impossibility of (1) for integers u, v, w not divisible by the odd prime n was proved by Sophie Germain for every odd prime $n < 100$, by Legendre for $n < 200$; the limit was extended to 223 by Maillet‡ and to 257 by Mirimanoff.§ In the present paper I establish the result for $n < 1700$.

2. Suppose that equation (1) has integral solutions u, v, w , without a common divisor, and no one a multiple of the odd prime n . Since

$$(-u)^n = v^n + w^n = (v + w) \phi, \quad \phi(v, w) = v^{n-1} - v^{n-2}w + \dots + w^{n-1},$$

* See H. J. S. Smith's *Report*, part II., § 61, British Association, 1860, pp. 148-152; *Collected Math. Papers*, Vol. I., p. 131; *l'Intermédiaire des Mathématiciens*, 1895, p. 117, p. 359; 1905, p. 11; 1906, p. 99.

† *Mém. Acad. Sc. Inst. France*, VI. (1823); *Théorie des Nombres*, second supplement (1825).

‡ *Assoc. Française*, St. Etienne, 1897, part II., p. 161.

§ *Crelle*, 128 (1905), p. 55.

a common prime factor of $v + w$ and ϕ would divide u and

$$\phi(v, -v) = nv^{n-1},$$

contrary to our assumptions. Hence $v + w$ and ϕ are exact n^{th} powers whose product is $(-u)^n$. Hence, by the symmetry of (1), we may set

$$(2) \quad \begin{cases} v + w = a^n, & \phi(v, w) = \alpha^n, & u = -a\alpha; \\ w + u = b^n, & \phi(w, u) = \beta^n, & v = -b\beta; \\ u + v = c^n, & \phi(u, v) = \gamma^n, & w = -c\gamma. \end{cases}$$

It follows that

$$(3) \quad 2u = b^n + c^n - a^n, \quad 2v = a^n + c^n - b^n, \quad 2w = a^n + b^n - c^n.$$

3. THEOREM. *If there exists an odd prime p such that the congruence*

$$(4) \quad \xi^n + \eta^n + \zeta^n \equiv 0 \pmod{p}$$

has no set of integral solutions ξ, η, ζ , each not divisible by p , and such that n is not the residue of the n^{th} power of any integer ρ ,

$$(5) \quad n \not\equiv \rho^n \pmod{p},$$

then equation (1) has no set of integral solutions each prime to n .

For, if u, v, w are integers satisfying equation (1), they satisfy the congruence (4). Hence one of them, say u , is divisible by p . By (3),

$$b^n + c^n + (-a)^n \equiv 0 \pmod{p}.$$

Hence one of the integers a, b, c is divisible by p . But if b were divisible by p , then, by (2), $v = -b\beta$ would be divisible by p , and hence, by (1), also w would be divisible by p , whereas u, v, w have no common factor. Similarly c is not divisible by p . Hence

$$a \equiv 0, \quad u \equiv 0, \quad w \equiv -v, \quad \phi(u, v) \equiv v^{n-1}, \quad \phi(v, w) \equiv nv^{n-1} \pmod{p}.$$

Thus, by (2), $\gamma^n \equiv v^{n-1}$, $\alpha^n \equiv nv^{n-1}$. Hence $n\gamma^n \equiv \alpha^n \pmod{p}$. By the final equation (2), γ is prime to p . Hence we may determine an integer γ_1 such that $\gamma\gamma_1 \equiv 1 \pmod{p}$. Hence $n \equiv (\alpha\gamma_1)^n \pmod{p}$, contrary to (5).

4. We note certain restrictions on the selection of the prime p required in § 3. First, n must divide $p - 1$. For, if not, we may determine integers s and t such that

$$1 = sn + t(p - 1).$$

Then any integer r is the residue (mod p) of an n^{th} power, since

$$r = (r^e)^n \cdot (r^e)^{p-1} \equiv (r^e)^n \pmod{p}.$$

Thus, to obtain solutions of (4) each not divisible by p , we may assign any integral values not $\equiv 0$ to η and ζ , such that $\eta^n + \zeta^n$ is not $\equiv 0$; then $-\eta^n - \zeta^n$ is the residue of the n^{th} power of some integer ξ .

Next, $3n$ must not divide $p-1$. For, if so, there exists an integer k such that

$$\frac{k^{3n} - 1}{k^n - 1} = k^{2n} + k^n + 1 \equiv 0 \pmod{p},$$

so that (4) has the set of solutions $\xi = 1, \eta = k, \zeta = k^2$. Hence

$$(6) \quad p = mn + 1 \quad (m \text{ an even integer prime to } 3).$$

5. Let $m = 2\mu$ be any even integer prime to 3, and n any* integer > 1 (not necessarily prime), such that $mn + 1$ is a prime p . Suppose that, contrary to the desired result, congruence (4) has integral solutions no one divisible by p . Determine τ so that $\tau\xi \equiv 1 \pmod{p}$ and multiply (4) by τ^n . The resulting congruence of the form

$$(7) \quad 1 + r^n \equiv s^n \pmod{p}$$

has integral solutions r not $\equiv 0$, s not $\equiv 0$. By Fermat's theorem

$$r^{mn} \equiv 1, \quad s^{mn} \equiv 1 \pmod{p = mn + 1}.$$

Let $z = r^n$. Then the congruences

$$(8) \quad z^m \equiv 1, \quad (1 + z)^m \equiv 1 \pmod{p}$$

have a common integral solution z . Conversely, from an integral solution z of the pair (8), we may derive integral solutions r, s , each not $\equiv 0$, of (7), and hence integral solutions, each not $\equiv 0$, of (4).

6. Let $z \equiv a$ be a common solution of congruences (8) Then

$$(9) \quad a, \quad \frac{1}{a}, \quad -1 - a, \quad \frac{1}{-1 - a}, \quad -1 - \frac{1}{a}, \quad \frac{-a}{1 + a}$$

* The advantage of allowing n to be general is that the work for a given m will be applicable in the discussion for m' , a multiple of m .

are common solutions of (8). They form a closed system under the operations of taking the reciprocal and of subtracting from -1 . Also a may be replaced by any one of the six by a succession of these operations. Hence, if any two are congruent, then a is congruent to one of the six. Since a is not $\equiv 0$, or -1 , while $a^2 + a + 1$ is not $\equiv 0$ (for $p \neq 3$), equalities arise only when $a \equiv 1, -\frac{1}{2}$, or -2 . To exclude these cases it is necessary and sufficient that

$$(10) \quad 2^m - 1 \text{ is not divisible by } p.$$

Making this assumption here (see § 8), we have six incongruent numbers (9). An equation $z^6 + \dots = 0$ having these six numbers as roots must be unaltered when z is replaced by $1/z$ or by $-1 - z$. Hence it has the form

$$(11) \quad z^6 + 3z^5 + bz^4 + (2b - 5)z^3 + bz^2 + 3z + 1 = 0.$$

Its left member must be a factor, modulo p , of

$$(12) \quad \frac{z^{2\mu} - 1}{z^2 - 1} = z^{\mu-1} \left(z^{\mu-1} + z^{\mu-3} + \dots + \frac{1}{z^{\mu-3}} + \frac{1}{z^{\mu-1}} \right),$$

no one of the numbers (9) being ± 1 . Let

$$(13) \quad x = z + 1/z, \quad \beta = b - 3.$$

The left member of (11) and the second factor* in (12) become, respectively

$$(14) \quad x^3 + 3x^2 + \beta x + 2\beta - 5,$$

$$(15) \quad x^{\mu-1} - (\mu-2)x^{\mu-3} + \frac{(\mu-3)(\mu-4)}{1 \cdot 2} x^{\mu-5} - \dots \\ + (-1)^s \frac{(\mu-s-1)(\mu-s-2)\dots(\mu-2s)}{1 \cdot 2 \dots s} x^{\mu-2s-1} + \dots$$

For μ even, the factor x of (15) may be dropped. Indeed, if $z + 1/z \equiv 0$ for a root of (11), then $z^2 \equiv -1$ for a common root of congruences (8). Then

$$(1+z)^2 \equiv 2z, \quad 1 \equiv (1+z)^{2\mu} \equiv 2^\mu z^\mu \pmod{p}.$$

Squaring, we get $1 \equiv 2^{2\mu} \equiv 2^m$, contrary to (10). Set

$$(16) \quad \text{function (15)} = f(x^2) \text{ or } xf(x^2), \text{ according as } \mu \text{ is odd or even.}$$

Hence, for x arbitrary, $f(x^2)$ is divisible by (14), modulo p . Changing† x to $-x$, we conclude that $f(x^2)$ is divisible by

$$(17) \quad x^3 - 3x^2 + \beta x - (2\beta - 5).$$

* Serret, *Algèbre*, I., p. 240 and bottom of p. 241.

† We are virtually enquiring when the numbers (9) and their negatives give twelve incongruent roots of $z^m \equiv 1$.

It was shown that x is not a factor of (14). Hence a common divisor of (14) and (17) divides $x^2 + \beta$ and $3x^2 + 2\beta - 5$, and thus exists only when $\beta + 5 \equiv 0$. For $\beta + 5$ not $\equiv 0$ $f(x^2)$ is therefore divisible by the product of (14) and (17), viz.

$$(18) \quad x^6 + (2\beta - 9)x^4 + (\beta^2 - 12\beta + 30)x^2 - (2\beta - 5)^2.$$

For $\beta \equiv -5$, (14) has the factors $x^2 - 5$ and $x + 3$. To exclude this case it suffices* to assume (see §9) that

$$(19) \quad f(x^2) \text{ is not } \equiv 0 \pmod{p} \text{ for } x \equiv -3.$$

7. We consider the question, arising only for $\mu \geq 7$, of the divisibility of $f(x^2)$ by (18), modulo p . By hypothesis, μ is prime to 3.

For $\mu = 7$, $f(x^2) = x^6 - 5x^4 + 6x^2 - 1$. This must be congruent to (18). Hence $p = 2$, contrary to hypothesis.

For $\mu = 8$, $f(x^2) = x^6 - 6x^4 + 10x^2 - 4$. Thus $p = 17$. But $n > 1$.

For $\mu = 10$, $f(x^2) = x^8 - 8x^6 + 21x^4 - 20x^2 + 5$. In the remainder from the division by (18), the coefficients of x^4 , x^2 , x^0 are

$$3\beta^2 - 8\beta, \quad 2\beta^3 - 21\beta^2 + 52\beta - 25, \quad 5 - (2\beta - 1)(2\beta - 5)^2.$$

For $3\beta \equiv 8$, the second gives $p = 61$, while the third is a multiple of 61. The value of $p = 61$ is exceptional, since (8) are both satisfied for $z \equiv 27$, the indices of 27 and 28 being multiples of $n = 3$.

For $\mu = 11$, $f(x^2) = x^{10} - 9x^8 + 28x^6 - 35x^4 + 15x^2 - 1$. In the remainder the coefficients of x^4 , x^2 , x^0 are

$$A = -4\beta^3 + 19\beta^2 - 10\beta - 28,$$

$$B = -3\beta^4 + 34\beta^3 - 120\beta^2 + 106\beta + 75,$$

$$C = -1 - (2\beta - 5)^2(-3\beta^2 + 6\beta - 2).$$

Then $16B + (79 - 12\beta)A = -23D$, where

$$D = 13\beta^2 - 54\beta + 44.$$

Since $n > 1$, $p \neq 23$. Thus $D \equiv 0$. Now $13A + 4\beta D$ gives

$$31\beta^2 + 46\beta - 364 \equiv 0.$$

* It is necessary that (19) should hold, if n be odd. Indeed, the corresponding factors of (11) are $z^2 \pm z - 1$, $z^2 + 3z + 1$. If one of these vanishes, so that

$$z(z+1) \equiv 1, \quad z^2 \equiv z+1, \quad \text{or} \quad (z+1)^2 \equiv -z,$$

and if $z^m \equiv 1$, then $(z+1)^m \equiv 1$, so that (8) would have a common root. In the third case, we have $(z+1)^{2m} \equiv 1$ and $(z+1)^{nm} \equiv 1$, by Fermat's theorem. Since n is odd, $(z+1)^m \equiv 1$.

Eliminating β^2 , we get $16(142\beta - 381) \equiv 0$, whence

$$142\beta \equiv 381.$$

Substituting this in $(142)^2 D = 0$, we get $-13^3 \cdot 67$. Hence $p = 67$, $n = 3$. The value $p = 67$ is exceptional, since (8) are both satisfied for $z \equiv 8$, the indices of 8 and 9 being multiples of $n = 3$.

For $\mu = 13$, $f(x^3) = x^{12} - 11x^{10} + 45x^8 - 84x^6 + 70x^4 - 21x^2 + 1$. In the remainder from the division by (18), the coefficients of x^4, x^2, x^0 are

$$A = 5\beta^4 - 32\beta^3 + 30\beta^2 + 102\beta - 124,$$

$$B = 4\beta^6 - 49\beta^4 + 204\beta^3 - 213\beta^2 - 422\beta + 684,$$

$$C = 1 - (2\beta - 5)^2(4\beta^3 - 13\beta^2 - 4\beta + 26).$$

From $\frac{1}{3}(4\beta A - 5B)$, we get

$$D = 39\beta^4 - 300\beta^3 + 491\beta^2 + 538\beta - 1140 = 0.$$

From* $\frac{1}{5}\{(5\beta - 7)D + (105 - 39\beta)A\}$ and $39A - 5D$, we get

$$E = 5\beta^3 - 315\beta^2 + 1216\beta - 1008 = 0,$$

$$F = 252\beta^3 - 1285\beta^2 + 1288\beta + 864 = 0.$$

The constant terms of E and F are $-7 \cdot 144$ and $6 \cdot 144$. Hence

$$\beta^{-1}(6E + 7F) = G = 1794\beta^2 - 10885\beta + 16312 = 0,$$

$$A - \beta E - F = H = 31\beta^3 + 99\beta^2 - 178\beta - 988 = 0,$$

$$F - 7E - 7H = J = 227\beta^2 - 5978\beta + 14836 = 0,$$

$$40H - 5F + 4E = K = 9125\beta^2 - 8696\beta - 47872 = 0.$$

$$G - 3J = 7 \cdot 53L, \quad L = 3\beta^2 + 19\beta - 76.$$

For $p = 53$, $z \equiv 6$ is a solution of congruences (8), since the indices of 6 and 7 are even. Let, henceforth, $p \neq 53$. Then $L \equiv 0$. Again, for $p = 79$, $z \equiv 14$ is a solution of congruences (8), since the indices of 14 and 15 are multiples of $n = 3$. Reducing modulo 79, we readily find that in $2L - 19J + 3K$ the coefficients of β^2, β, β^0 are multiples of 79. For the original values, we get

$$2L - 19J + 3K = 4 \cdot 79M, \quad M = 73\beta^2 + 277\beta - 1347.$$

Let $p \neq 79$. Then $M \equiv 0$. The determinant of the coefficients in L, J, M equals

$$270777 = 3 \cdot 13 \cdot 53 \cdot 131.$$

* We seek equations with small coefficients.

For $p=131$, $z \equiv 18$ is a solution of (8), since the indices of 18 and 19 are multiples of $n=5$.

8. Consider condition (10) for $m=2\mu$, $\mu \leq 13$, μ prime to 3. For these cases the prime factors of $2^m - 1$ are as follows:

m	$2^m - 1$	m	$2^m - 1$	m	$2^m - 1$
2	3	10	$3 \cdot 11 \cdot 31$	20	$3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$
4	$3 \cdot 5$	14	$3 \cdot 43 \cdot 127$	22	$3 \cdot 23 \cdot 89 \cdot 683$
8	$3 \cdot 5 \cdot 17$	16	$3 \cdot 5 \cdot 17 \cdot 257$	26	$3 \cdot 8191 \cdot 2731$

For the seven values of $m \geq 8$, there is a prime factor of the form $mn+1$, $n > 1$. Of these, n is an odd prime only for*

(20) $m=10$, $31=3m+1$; $m=14$, $43=3m+1$; $m=22$, $683=31m+1$.

There occurs no factor $mn+1$, $n=2^k N$, $k > 0$, N an odd prime.

9. Consider condition (19) for $\mu \leq 13$. We have

μ	2	4	5	7	8	10	11	13
$f(9)$	1	7	$5 \cdot 11$	$13 \cdot 29$	$7 \cdot 47$	$5 \cdot 11 \cdot 41$	$89 \cdot 199$	$233 \cdot 521$

The factors of the form $mn+1$ ($m=2\mu$) have $n=1, 2, 4, 9, 20$ (the last for $\mu=13$). In particular, n is never an odd prime.

10. The preceding results lead to the following:—

THEOREM. *If n and $p=mn+1$ are odd primes, m being prime to 3 and $m \leq 26$, then the congruence*

$$\xi^n + \eta^n + \zeta^n \equiv 0 \pmod{p}$$

has no integral solutions each prime to p , except for

(21) $n=3, m=10, 14, 20, 22, 26$; $n=5, m=26$; $n=31, m=22$.

If $n=2^k N$, $k > 0$, N an odd prime, and $m < 26$, there are no integral solutions prime to p .

11. A direct treatment will be given for certain larger values of $m=2\mu$. We readily make a reduction of the number of cases to be treated (see also § 16). By (8),

$$z^\mu \equiv \pm 1, (1+z)^\mu \equiv \pm 1 \pmod{p}.$$

If both signs are minus, $Z=1/z$ satisfies

$$(22) \quad Z^\mu \equiv -1, (1+Z)^\mu \equiv +1,$$

and, for μ odd, $r=-1-z$ satisfies

$$(23) \quad r^\mu \equiv +1, (1+r)^\mu \equiv +1.$$

* The only additional cases for $m \leq 64$ in which $2^m - 1$ has a prime factor $mn+1$, n an odd prime, are $m=50$, $251=5m+1$; $m=52$, $157=3m+1$, $1613=31m+1$; $m=58$, $1103=19m+1$. For $m=128$ there are two such factors.

For μ even, we replace Z by $-1-z$ in (22) and get

$$(24) \quad z^\mu \equiv +1, (1+z)^\mu \equiv -1.$$

Hence for any μ it suffices to consider (23) and (24).

For μ even, (23) is the original problem for a smaller value of m ; hence there is an advantage in the selection of values of m which are multiples of 4. By § 10, there is no integral solution of (23) for $p = \mu(2n) + 1$ when $\mu = 14, 16$, or 20 ; hence for $m = 28, 32, 40$, it remains to discuss (24) alone.

12. Let $m = 28$. We discuss (24) for $\mu = 14$. Let

$$(25) \quad x = z + 1/z, (1+z)^2/z = x + 2.$$

First let $z^7 \equiv -1$. Then $(z^7 + 1)/(z + 1) \equiv 0$ gives

$$(26) \quad x^3 \equiv x^2 + 2x - 1.$$

By (25), $(x + 2)^7 \equiv 1$. Employing (26), we get

$$(27) \quad \begin{cases} (x+2)^2 \equiv 7(x^2+2x+1), & (x+2)^4 \equiv 7(5x^2+7x+1), \\ (x+2)^6 \equiv 7(22x^2+25x-3), & (x+2)^8 \equiv 49(13x+13x-4). \end{cases}$$

Now $x + 2$ is not $\equiv 1$. Hence

$$\frac{1}{2} \sum_{i=0}^6 (x+2)^i \equiv A \equiv 417x^3 + 440x - 98 \equiv 0.$$

From xA eliminate x^3 by (26). Subtracting the result from $2A$, we get

$$B \equiv -23x^2 + 144x + 221 \equiv 0.$$

Also, $xB \equiv C \equiv 121x^2 + 175x + 23 \equiv 0$. The determinant of the coefficients of A, B, C equals $-645569 = -29 \cdot 113 \cdot 197$. For these factors, $n = 1, 4, 7$, respectively. We note that $19^7 \equiv -1, 20^{14} \equiv -1 \pmod{197}$.

Next, let $z^7 \equiv +1$. If $z \equiv 1$, (24) requires that p divide $2^{14} + 1$, so that $p = 29$ or 113 . For z not $\equiv 1$, $(z^7 - 1)/(z - 1) \equiv 0$ becomes

$$(28) \quad x^3 \equiv -x^2 + 2x + 1.$$

Also $(x + 2)^7 \equiv -1$ by (25). Now

$$(29) \quad \begin{cases} (x+2)^3 \equiv 5x^2+14x+9, & (x+2)^4 \equiv 19x^2+47x+23, \\ (x+2)^6 \equiv 221x^2+507x+196, & (x+7)^7 \equiv 728x^2+1652x+613. \end{cases}$$

From the last power we get $A = 364x^2 + 826x + 307 \equiv 0$. To xA we apply (28), then subtract A , and get

$$B = 98x^2 + 209x + 57 \equiv 0.$$

Let $C = 4B - A$. To xC we apply (28) and get

$$D = 18x^2 + 23x - 28 \equiv 0.$$

The determinant in B, C, D equals $4 \cdot 13 \cdot 29^2$.

Hence for n and $28n+1=p$ both odd primes, congruence (4) has no integral solutions prime to p , except when $n=7$.

13. For $m=32$, conditions (24) may be written in the form

$$(30) \quad z^{16} \equiv 1, \quad u^{16} \equiv -1, \quad u \equiv 1+z \pmod{p}.$$

(I). First, let $z^8 \equiv -1$. Then $(u^8/z^4)^2 \equiv 1$, $u^8 \equiv \mp z^4$, viz.

$$8z^6 + 28z^5 + 56z^4 + (70 \pm 1)z^3 + 56z^2 + 28z + 8 \equiv 0.$$

Let $x = z + 1/z$. The latter and $z^8 + 1 \equiv 0$ become

$$A = 8x^3 + 28x^2 + 32x + 14 \pm 1 \equiv 0,$$

$$B = x^4 - 4x^2 + 2 \equiv 0.$$

Form*

$$xA - 8B = C = 28x^3 + 64x^2 + (14 \pm 1)x - 16 \equiv 0,$$

$$7A - 2C = D = 68x^3 + (196 \mp 2)x + 130 \pm 7 \equiv 0,$$

$$17A - 2xD = E = (84 \pm 4)x^2 + (284 \pm 14)x + 238 \pm 17 \equiv 0,$$

$$17E - (21 \pm 1)D = F = (714 \mp 392)x + 1309 \pm 12 \equiv 0.$$

(I₁). For the upper signs,

$$F = 322x + 1321 \equiv 0, \quad 34xF - 161D = 13680x - 22057 \equiv 0.$$

The determinant of the coefficients equals $-2 \cdot 17^2 \cdot 97 \cdot 449$. The last two factors equal $3 \cdot 32 + 1$ and $14 \cdot 32 + 1$. The only case (for n prime) is thus $p=97$; then $18^8 \equiv 19^{16} \equiv -1 \pmod{97}$.

(I₂). For the lower signs,

$$F = 1106x + 1297 \equiv 0, \quad 553D - 34xF = 65396x + 68019 \equiv 0.$$

The determinant equals $-2 \cdot 17^2 \cdot 47 \cdot 353$. For

$$p = 353 = 11 \cdot 32 + 1,$$

we have $x \equiv 40$, $z \equiv 100$, $z^8 \equiv -1$, $u^8 \equiv z^4$.

(II). Next, let $z^4 \equiv -1$. Then $u^8 \equiv \mp z^2$. Hence

$$(1/z)^4 \equiv -1, \quad \{1 + (1/z)\}^8 \equiv u^8/z^8 \equiv \mp z^2 \equiv \pm (1/z)^2.$$

Hence it suffices to treat the case $u^8 \equiv +z^2$. Then $u^4 \equiv \mp z$,

$$(1+z)^4 \mp z \equiv 0, \quad -6z \equiv 4z^2 + 4 \pm 1.$$

Squaring the latter, we get $(4 \pm 8)z^2 = 1 \pm 8$. Squaring, we find that $p=97$ or 193 . But the latter is $32n+1$, $n=6$.

(III). For $z^2 \equiv -1$, we have $u^2 \equiv 2z$, $u^4 \equiv -4$, so that $u^{16} \equiv -1$ gives $p=257=32n+1$, $n=8$.

* A shorter method would be to transpose the terms x^4, x in A , square, and apply B . Finally, the method of §17 has the advantage of avoiding the factorization of large numbers.

(IV). Finally, let $z^2 \equiv 1$. Since z is not $\equiv -1$ by (30), we have $z \equiv 1, u \equiv 2$,

$$p = 2^{16} + 1 = 65537 = 32n + 1, \quad n = 2^{11}.$$

Hence for n and $32n + 1 = p$ both odd primes, congruence (4) has no integral solutions prime to p , except when $n = 3$.

14. For $m = 40$, congruences (24) become

$$z^{20} \equiv 1, \quad u^{20} \equiv -1, \quad u = 1 + z \pmod{p}.$$

(I). First, let $z^{10} \equiv -1$. Then $u^{10} \equiv \mp z^5$, viz.

$$10z^8 + 45z^7 + 120z^6 + 210z^5 + (252 \pm 1)z^4 \\ + 210z^3 + 120z^2 + 45z + 10 \equiv 0.$$

For $x = z + 1/z$, the latter becomes

$$A = 10x^4 + 45x^3 + 80x^2 + 75x + 32 \pm 1 \equiv 0,$$

so that x is not $\equiv 0$; while $z^{10} + 1 \equiv 0$ becomes $xB \equiv 0$, where

$$B = x^4 - 5x^2 + 5 \equiv 0.$$

Transpose* the terms x^3 and x of A , square, and apply B ; then

$$x^2 (\pm 260 - 55) + 200 \mp 36 \equiv 0.$$

Multiply the latter by $-x^2$ and apply B ; then

$$x^2 (75 \mp 1264) - 275 \pm 1300 \equiv 0.$$

For the upper and lower signs the determinant equals, respectively,

$$41^2 \cdot 241, \quad 281 \cdot 641.$$

The primes are $40n + 1$, $n = 1, 6, 7, 16$. For $p = 281$, congruences (8) have the common root $z \equiv 213$.

(II). Next, let $z^5 \equiv -1$. Applying (25), we get

$$x^2 \equiv x + 1, \quad (x + 2)^{10} \equiv -1.$$

By the first, $(x + 2)^2 = 5(x + 1)$, $(x + 2)^{10} \equiv 5^5(55x + 34)$. We obtain at once two linear equations with the determinant $41 \cdot 281 \cdot 881$.

(III.) Finally, let $z^5 \equiv +1$. If $z \equiv 1$, then $u \equiv 1$ and p divides

$$2^{20} + 1 = 17 \cdot 61681.$$

* If we eliminate in turn x^4, x^3, x^2 , we have a longer analysis and introduce extraneous factors 2 and 17.

For the second, $n=1542$ is not a prime. Hence z is not $\equiv 1$. Thus

$$x^2 \equiv -x + 1, \quad (x+2)^5 \equiv 55x + 89,$$

$$0 \equiv 1 + (x+2)^{10} \equiv 3 \cdot 41 (55x + 89).$$

Since $p \neq 3$, $p \neq 41$, this contradicts the previous congruence.

For n and $40n+1=p$ both odd primes, congruence (4) has no integral solutions prime to p , except when $n=7$.

15. For $m=56$ we are to consider* $z^{28} \equiv 1$, $u^{28} \equiv -1$.

First, let $z^{14} \equiv -1$. Then $u^{14} \equiv \mp z^7$. By (25),

$$(31) \quad (x+2)^7 \equiv \mp 1.$$

No prime factor of $2^7 \pm 1$ is of the form $56n+1$. Hence $x \neq 0$. Thus, from $z^7 + 1/z^7 = 0$, we get

$$(32) \quad x^6 \equiv 7x^4 - 14x^2 + 7.$$

We expand (31), reduce to fifth degree by (32), carry the even powers to the second member, square, and again reduce by (32). Deleting the factor 2, we get

$$(33) \quad (364 \pm 378)x^4 + (-2282 \pm 476)x^2 + 3415 \pm 226 \equiv 0.$$

We multiply (33) by x^2 and eliminate x^6 by (32). With the new equation we proceed similarly. In the resulting three equations the determinant of the coefficients of x^4 , x^2 , x^0 equals

$$281 \cdot 449 \cdot 617 \cdot 953, \quad 113^3 \cdot 337 \cdot 7001,$$

for the upper and lower signs, respectively.

Next, let $z^7 \equiv -1$. Then (26) holds, and

$$(34) \quad [(x+2)^{14} + 1] \div [(x+2)^2 + 1] \equiv 0,$$

the denominator being zero for no admissible modulus. Continuing the series of powers (27), we get

$$(x+2)^8 \equiv 49(204x^2 + 179x - 94),$$

$$(x+2)^{10} \equiv 7^3(435x^2 + 362x - 225),$$

$$(x+2)^{12} \equiv 7^4(910x^2 + 741x - 491).$$

After deleting the factor 2, (34) becomes

$$A = 1022549x^3 + 831577x - 553061 \equiv 0.$$

* For both signs plus, there is no modulus $56v+1$, v an odd prime (§ 12).

We form $2A - xA = B$, reducing by (26); then $2B - xB = C$. To obtain smaller numbers we use the equations $C, A - 5B, B - 4C$. The determinant is

$$\begin{vmatrix} 19855 & 43863 & 23826 \\ 67689 & -24008 & -135196 \\ 111552 & -4335 & -178877 \end{vmatrix} = 337 \cdot 6329 \cdot 12713.$$

Finally, let $z^7 \equiv +1$. Then we have (28), (29), and

$$(x+2)^8 \equiv 2380x^2 + 5373x + 1954,$$

$$(x+2)^{10} \equiv 25213x^2 + 56714x + 20329,$$

$$(x+2)^{12} \equiv 266110x^2 + 598091x + 213669.$$

Now $(x+2)^2 \equiv -1$ and (28) have a common solution, viz. $x \equiv -11$, if, and only if, $p = 41$, an excluded modulus. Hence we apply (34), delete the factor 2, and obtain

$$E = 121537x^2 + 273143x + 97559 \equiv 0.$$

We form $xE - E = F, E - 4F = G, xG - G = H, xH = J$. The determinant of the coefficients in $J, 4J - G$, and $H - J$ is

$$\begin{vmatrix} 325 & 936 & 661 \\ 39 & 561 & 997 \\ 336 & 50 & -1047 \end{vmatrix} = 113 \cdot 281 \cdot 673.$$

The various primes in the three cases are

$$113, 281, 337, 449, 617, 673, 953, 6329, 7001, 12713.$$

These are of the form $56n + 1$, with respectively

$$n = 2, 5, 6, 8, 11, 12, 17, 113, 125, 227.$$

For n and $56n + 1 = p$ both odd primes, congruence (4) has no integral solutions prime to p , except when $n = 5, 11, 17, 113, 227$.

16. The case in which m is a power of 2 offers certain advantages, since we may separate cases and largely avoid the factorization of high numbers, and since the treatment (§ 20) of condition (5) is then relatively simple. Let $m = 2^r$. If we have already treated the case 2^{r-1} , it remains to consider (see § 11)

$$(35) \quad z^{2^{r-1}} \equiv 1, \quad u^{2^{r-1}} \equiv -1, \quad u = 1 + z.$$

(I). First, let $z^{2r-2} \equiv -1$. Then

$$(36) \quad u^{2r-2} \equiv \mp z^{2r-3}.$$

(I₁). Let the sign be minus. Then

$$u^{2r-2} \equiv 1/z^{2r-3}, \quad u^{2r-3} \equiv \mp 1/z^{2r-4},$$

$$(1 + 1/z)^{2r-3} \equiv \mp 1/z^{2r-4+2r-3} = \pm z^{2r-4} = \pm 1/(1/z)^{2r-4}.$$

It therefore suffices to treat the case of the lower sign. Then

$$(37) \quad u^{2r-4} \equiv \mp 1/z^{2r-5}.$$

Thus, for $r=5$, $u^2 \equiv \pm 1/z$. For $r=6$, there is the additional case

$$u^4 \equiv -1/z^2 \equiv z^{14}, \quad u^2 \equiv \mp z^7.$$

For $r=7$, the cases are $u^2 \equiv \pm 1/z, \pm z^{15}, \pm z^7, \pm 1/z^9$.

(I₁). Let the sign in (36) be plus. Then

$$(38) \quad u^{2r-3} \equiv \mp z^{2r-4}.$$

For the lower sign, the square root gives (for $r > 5$)

$$(39) \quad u^{2r-5} \equiv \mp z^{2r-6},$$

or else $u^{2r-4} \equiv -z^{2r-5} \equiv 1/z^{2r-2-2r-5}, \quad u^{2r-5} \equiv \mp 1/z^{2r-3-2r-6},$

$$\{1 + (1/z)\}^{2r-5} \equiv \mp 1/z^{2r-3+2r-6} \equiv \pm z^{2r-2-2r-3-2r-6} \equiv \pm 1/(1/z)^{2r-3-2r-6},$$

so that we may drop the minus case and get

$$(40) \quad u^2 \equiv 1/z^7 (r=6), \quad u^{2r-6} \equiv \mp 1/z^{2r-4-2r-7} (r > 6).$$

Similarly, if $r > 6$, (39) may be replaced by the cases

$$(41) \quad u^{2r-6} \equiv \mp z^{2r-7}, \quad u^{2r-6} \equiv 1/z^{2r-8-2r-7}.$$

For the upper sign (38), we obtain two equivalent cases, one of them giving

$$(42) \quad u^2 \equiv 1/z^3 (r=5), \quad u^{2r-5} \equiv \mp 1/z^{2r-4-2r-6} (r > 5).$$

(II). For $z^{2r-1} \equiv -1$ ($l > 2$), $u^{2r-2} \equiv \mp z^{2r-1}$. The two cases are equivalent under reciprocation.

Under (I) the case $u^2 \equiv z$ is excluded by raising $1+z \equiv -z^2$ to the power $2r-1$.

17. For $m=2^5$, the cases under $z^8 \equiv -1$ are $u^2 \equiv 1/z, -1/z, 1/z^3, -z$. The corresponding moduli are 449, 97, 353, 47. For case (II) see §13.

18. For $m=2^6$, the cases* under $z^{16} \equiv -1$ and the moduli are

$u^2 =$	$1/z$	$-1/z$	z^7	$-z^7$	$1/z^7$	$1/z^3$	$-1/z^3$
	204353	7937	14657	193 257	193·641	65921	27457

These primes are of the form $64n+1$ with, respectively,

$n = 31 \cdot 103, 128, 229, 3, 4, 3, 10, 1030, 3 \cdot 11 \cdot 13$.

For $u^2 \equiv -z, p=2207$ is not of the form $64+1$.

For $z^8 \equiv -1, u^4 = z$ gives $p=48449$ ($n=757$), $u^4 = -z$ gives $p=33409$ ($n=522$), $u^4 = 1/z^3$ gives $p=63361$ ($n=990$), $u^4 = -1/z^3$ gives $p=7937$ ($n=128$).

For $z^4 \equiv -1, u^8 \equiv \mp z$ give 21569 ($n=337$), 15809 ($n=13 \cdot 19$).

For $z^2 \equiv -1, u^2 \equiv 2z, -1 \equiv u^{32} \equiv 2^{16}, p=65537$ ($n=2^{10}$).

For $z \equiv 1, 2^{32} \equiv -1, p=641$ ($n=10$), $2^7 \cdot 52347 + 1$ (n even).

To illustrate the detailed work, let $z^{16} \equiv -1, u^2 \equiv \mp z^7$. Then

$$(z^7 \pm 2z)^2 \equiv (z^2 + 1)^2, (z^{14} + 2z^2)^2 \equiv (1 + z^4 \mp 4z^8)^2,$$

$$[(\pm 8 - 1) z^{12} + 2z^4]^2 \equiv [(1 \mp 8) z^8 - 11]^2,$$

$$(-39 \mp 160) z^8 \equiv 52 \pm 48, (39 \pm 160)^2 + (52 \pm 48)^2 \equiv 0.$$

The prime values of n are 3, 229, 337, 757. We must also include $n=7$ from 449 for the case delegated to $m=32$.

For n and $64n+1=p$ both odd primes, congruence (4) has no integral solutions prime to p , except for $n=3, 7, 229, 337, 757$.

19. We pass to the consideration of condition (5), when n and $p=mn+1$ are odd primes. The condition fails only when $n^m \equiv 1$. Since m is even and $mn \equiv -1$, the last condition is equivalent to

$$(43) \quad m^m \equiv 1 \pmod{p}.$$

Suppose first that m is a multiple of 4. Then the quadratic reciprocity law gives

$$(n/p) = (p/n) = +1, n^{\frac{1}{2}(p-1)} \equiv n^{\frac{1}{2}mn} \equiv 1 \pmod{p}.$$

But $n^m \equiv 1$. Hence, n being odd, $n^{\frac{1}{2}m} \equiv 1$ and

$$(44) \quad m^{\frac{1}{2}m} \equiv 1 \pmod{p},$$

for m a multiple of 4.

* To proceed independently of §16, we note that z belongs to the exponent 32, so that $u^2 \equiv z^a$ (a odd). For $Z=1/z, (1+Z)^2 = Z^{4-a}$. Hence a and $2-a$ lead to equivalent cases.

Hence for* $m = 20$, we seek the factors of $20^5 \pm 1$. By

$$(a^5 - b^5)/(a - b) = (a^2 + 3ab + b^2)^2 - 5ab(a + b)^2,$$

for $a = 20$, $b = 1$, we get

$$(20^5 - 1)/19 = 461^2 - 100 \cdot 21^2 = 251 \cdot 11 \cdot 61,$$

of which 61 alone is of the form $20n + 1$. Also

$$(20^5 + 1)/21 = 152381, \text{ a prime with } n = 19 \cdot 401.$$

For $m = 28$, we seek the factors of $28^7 \pm 1$. By

$$(a^7 + b^7)/(a + b) = (a + b)^6 - 7ab(a^2 + ab + b^2)^2,$$

for $a = 28$, $b = 1$, we get

$$(28^7 + 1)/29 = 29^6 - 7^2 \cdot 2^2 \cdot 813^2 = 13007 \cdot 35771,$$

neither prime having the form $28n + 1$. Also

$$(28^7 - 1)/27 = 113 \cdot 4422461,$$

the primes having $n = 4$, 157945, both composite.

For $m = 40$, we have

$$(40^{10} + 1)/(40^2 + 1) = 4237021 \cdot 1545781 = 241 \cdot 17581 \cdot 281 \cdot 5501,$$

the first step resulting from a formula† of Aurifeuille. Also

$$(40^5 + 1)/(40 + 1) = 11^2 \cdot 20641,$$

$$(40^5 - 1)/(40 - 1) = 2625641 = \text{prime.}^\ddagger$$

Finally, $40^2 + 1$ is prime. Hence, for $m = 40$, the only prime factor $mn + 1$, with n an odd prime, is 281.

20. For $m = 2^r$, $r \geq 3$, we may reduce further the exponent in (44). Since m is a multiple of 8, 2 is a quadratic residue of p , whence

$$(45) \quad 2^{2^{r-1}n} \equiv 1 \pmod{p = 2^r n + 1}.$$

* After this paper was in type, I learned from Lt.-Col. Cunningham that the cases $m = 20$ and 28 (and $m = 22$, now inserted in §21) were treated by Ed. Lucas, *Amer. Jour. Math.* vol. 1., p. 294. The former has under preparation a table of factors of $x^x \mp 1$ up to $x = 36$.

† For the various formulæ of Aurifeuille, see Bickmore, *Messenger of Mathematics*, vol. xxv., p. 1.

‡ Verified by Mr. E. B. Escott.

By (44), $2^{2^{r-1}r} \equiv 1 \pmod{p}$. Hence

$$(46) \quad 2^{2^{r-1}} \equiv 1,$$

unless r is divisible by the odd prime n .

For $r=3$, $2^3 \cdot 3 + 1$ is not prime. Hence $2^4 \equiv 1$, whereas $p \neq 3$, $p \neq 5$. For $r=4$, $2^8 \equiv 1$, whereas $p \neq 3$, $p \neq 5$, $p \neq 17$. For $r=5$, $2^5 \cdot 5 + 1 = 7 \cdot 23$, whence $2^{16} \equiv 1$, whereas $p=3, 5, 17$, or 257 is excluded. For $r=6$, $2^6 \cdot 3 + 1$ gives the exceptional prime $p=193$; the admissible factors of $2^{32}-1$ have $n=4$, $n=2^{10}$. For $m=2^r$ ($r \leq 6$) condition (43) holds only for $m=64$, $n=3$.

21. For $m=10$, we have

$$10^5 - 1 = 9 \cdot 41 \cdot 271, \quad 10^5 + 1 = 11 \cdot 9091,$$

the values of $n > 1$ being composite. For $m=14$,

$$\frac{1}{13}(14^7 - 1) = 8108731 = 1 + 14 \left\{ \frac{1}{13}(14^6 - 1) \right\},$$

$$\frac{1}{15}(14^7 + 1) = 7027567 = 1 + 14 \left\{ \frac{1}{15}(14^6 - 1) \right\},$$

the numbers being prime and the n 's composite.

For $m=22$, Lucas (*loc. cit.*) gives

$$\frac{1}{21}(22^{11} - 1) = 67 \cdot 353 \cdot A, \quad \frac{1}{23}(22^{11} + 1) = 89 \cdot B,$$

$$A = 1176469537, \quad B = 285451051007,$$

stating that A and B are primes. Lt.-Col. Cunningham believes that the last assertion is correct. In $A = 1 + 22n$, n is even. In $B = 1 + 22n$ I find that n is composite, viz. $n = 911 \cdot 14242643$. Hence for $m=22$, the only prime value of n is 3.

In order that our application shall be independent of the verification that A and B are primes, we note that $m=22$ is employed (§24) to exclude only the values 661, 1153, 1213, 1321, 1741 of n . For these five primes n , $p = 22n + 1$ is a prime. By the quadratic reciprocity law, $11^{11n} \equiv -1 \pmod{p}$. Since each $n \equiv 1 \pmod{4}$, we have $11n \equiv 3 \pmod{4}$, $p \equiv 7 \pmod{8}$, so that

$$2^{11n} \equiv +1, \quad 2^{2^{11n}} \equiv -1 \pmod{p}.$$

Hence if p divides $2^{2^2} - 1$, it divides $2^{2^{11}} + 1$. By trial, no one of the five values of p divides B and hence not $2^{2^2} - 1$.

22. For $m=56$, we have (Bickmore, *l.c.*, p. 39)

$$56^{14} + 1 = (56^2 + 1)\alpha\beta, \quad \alpha = 132342^2 + 35267^2, \quad \beta = 218554^2 + 54081^2.$$

By the *Canon Arithmeticus* of Jacobi, I find that for $p = 29, 113, 673$, and 701 , $56^{14} \equiv -1 \pmod{p}$. We multiply by p and then remove p^2 , by use of

$$(47) \quad (a^2 + b^2)(A^2 + B^2) = (aA \pm bB)^2 + (bA \mp aB)^2;$$

the latter numbers, for a suitable choice of sign, are both divisible by $a^2 + b^2$, when the latter is a prime factor of $A^2 + B^2$. Using (47) for the lower, upper, upper, and upper signs, respectively, we get

$$113\beta = (7^2 + 8^2)\beta = 113^2\gamma, \quad \gamma = 9710^2 + 18823^2,$$

$$29\gamma = (2^2 + 5^2)\gamma = 29^2\delta, \quad \delta = 3915^2 + 376^2,$$

$$673\alpha = (23^2 + 12^2)\alpha = 673^2\epsilon, \quad \epsilon = 3565^2 + 3894^2.$$

$$701\epsilon = (5^2 + 26^2)\epsilon = 701^2\lambda, \quad \lambda = 160^2 + 119^2.$$

Now $56^2 + 1 = 3137$ is prime, $\lambda = 39761$ prime (with $n = 710$), while* $\delta = 15468601$ has no prime factor of the form $56r + 1$. Hence $56^{14} + 1$ has no prime factor $56n + 1$ with n an odd prime.

23. From our results there follows the

THEOREM. *For n an odd prime, there is no set of integers u, v, w , each not divisible by n , such that*

$$u^n + v^n + w^n = 0,$$

whenever there exists a prime number p in the set

$$2n + 1, 4n + 1, 8n + 1, 10n + 1 (n \neq 3), 14n + 1 (n \neq 3), 16n + 1,$$

$$20n + 1 (n \neq 3), 22n + 1 (n \neq 3, n \neq 31),$$

$$26n + 1 (n \neq 3, n \neq 5; p \text{ not a factor of } 26^{13} \pm 1),$$

$$28n + 1 (n \neq 7), 32n + 1 (n \neq 3), 40n + 1 (n \neq 7),$$

$$56n + 1 (n \neq 5, 11, 17, 113, 227; p \text{ not a factor of } 56^7 \pm 1),$$

$$64n + 1 (n \neq 3, 7, 229, 337, 757).$$

24. For each of the 505 odd primes $n < 1700$, at least one of the preceding linear functions $mn + 1$ is a prime, except† for $n = 197, 223, 257, 383, 389, 457, 569, 751, 1373, 1399, 1531$.

* Lt.-Col. Cunningham informs me that δ is a prime, and that $\frac{1}{25}(56^7 - 1)$ and $\frac{1}{27}(56^7 + 1)$ have no divisors < 10000 .

† For the primes n between 1700 and 2000, the only exceptions are

$$n = 1709, 1787, 1933, 1949, 1993.$$

The least allowable value of m for which $mn + 1$ is prime are $m = 50, 104, 46, 68, 34$, respectively. These values of n have since been excluded.

This work was checked, after an interval of some weeks, by a different set of tables. It is most convenient to try, as the last resort, $m=22, 26, 56$.

The five values of $n < 2000$ excluded by $m=22$ (and by no other value of m) are given in §21. Those excluded by $m=26$ are $n=227, 317, 881, 887, 1181, 1217, 1847$; no one of the resulting values of $26n+1$ divides $26^{13} \pm 1$. Those excluded by $m=56$ are $n=1637$ and 1997 , $56n+1$ not dividing $56^7 \pm 1$.

$n=197$ is excluded by means of the prime $p=7487$, given by $m=38$ (Legendre), and another way by Maillet, *l. c.*, p. 160.

$n=223$ is excluded by Mirimanoff, *l. c.*, p. 55, by means of results due to Kummer.

$n=457$ and $n=751$ may be excluded by the method last cited. I find by use of the *Canon Arithmeticus* of Jacobi that

$$\sum_{h=77}^{114} h^{-3} \equiv 258 \pmod{457}, \quad \sum_{h=1}^{125} h^{-3} \equiv 54 \pmod{751}.$$

The remaining values of n I exclude by the method of Sophie Germain, employing the minimum allowable value of m for which $p=mn+1$ is prime.

$n=257, m=38, p=9767$. Since -2 is a primitive root of p , the residue 2134 of $(-2)^{257}$ modulo p belongs to the exponent m . By powering, we find that the numbers belonging to the exponent m are

1, 423, 2134, 2319, 2412, 2484, 2488, 2534, 2625, 3063,

3123, 3362, 3388, 3856, 4104, 4118, 4237, 4508, 4877,

and their negatives. No one of the 38 increased by unity gives one of the set. Also 257 does not occur among the 38. Hence $n=257$ is excluded.

$n=383, m=62, p=23747$. 2 is a primitive root of p , $2^{383} \equiv 6454$. The residues of the powers of the latter, mod p , are

1, 1039, 1216, 1341, 1878, 3196, 3206, 3936, 3988, 4015, 4833,

5020, 5903, 6342, 6346, 6454, 6491, 7774, 7887, 8172, 8560,

9052, 9159, 9642, 10718, 10870, 10906, 11285, 11376, 11419,

11554,

and their negatives. Hence 383 is excluded.

$n = 389$, $m = 38$, $p = 14783$. 2 is a primitive root of p ,
 $2^{389} \equiv -890$. The residues of the powers of the latter are

1, 52, 55, 890, 1744, 1931, 1990, 2704, 2724, 2860, 3025,
 3069, 3445, 3762, 4601, 5313, 5969, 6182, 7222,

and their negatives. Hence 389 is excluded.

$n = 569$, $m = 44$, $p = 25037$. 2 is a primitive root of p ,
 $2^{569} \equiv -6769$. The residues of the powers of the latter are

1, 478, 1488, 1651, 2112, 2405, 3062, 3151, 3232, 3958,
 4930, 5395, 5809, 6027, 6769, 7398, 8056, 9117, 10228,
 10888, 11490, 12006,

and their negatives. Hence 569 is excluded.

$n = 1373$, $m = 44$, $p = 60413$. 2 is a prime root of p . The
 residues of the powers of $2^{1373} \equiv -7283$ are

1, 525, 3649, 6053, 7283, 11234, 13199, 13990, 17491, 17556,
 18020, 22624, 23761, 24064, 24341, 25636, 26289, 26440,
 27561, 28531, 29447, 29595,

and their negatives. Hence 1373 is excluded.

$n = 1399$, $m = 70$, $p = 97931$. 2 is a primitive root of p .
 The residues of the powers of $2^{1399} \equiv 26401$ are

1, 3707, 4088, 6680, 7326, 7397, 11567, 13783, 14473, 14777,
 14909, 15051, 15300, 18198, 21743, 25089, 26401, 26613, 27820,
 29527, 30075, 30269, 30595, 31509, 34136, 34457, 34579,
 34910, 35438, 35964, 37874, 42547, 43195, 44519, 45112,

and their negatives. Hence 1399 is excluded.

$n = 1531$, $m = 52$, $p = 79613$. 2 is a primitive root. The
 residues of the powers of $2^{1531} \equiv 39379$ are

1, 1911, 2208, 3627, 4866, 5692, 8052, 10277, 10918, 11436,
 13367, 15795, 18871, 18984, 22063, 22087, 25064, 25104,
 25117, 29569, 29722, 31925, 32497, 32895, 34673, 39379,

and their negatives. Hence 1531 is excluded.

THEOREM. *For n an odd prime < 1700 , there is no set of
 integers u, v, w , each not divisible by n , such that*

$$u^n + v^n + w^n = 0.$$

ON SURFACES WHICH HAVE ASSIGNED FAMILIES OF CURVES AS THEIR LINES OF CURVATURE.

By *Prof. A. R. Forsyth.*

THE number of classes of surfaces, which are determined by the assignment of families of known curves as the lines of curvature of either system or of both systems, is comparatively small. The earliest known results are due to the investigations of Dupin; and owing to the fact that, in the surfaces considered by him, both sets of lines of curvature are circles, the surfaces are called cyclides: a wider significance was given to the title by the researches of Darboux. An important class of such surfaces has been discussed in the investigations of Serret, Bonnet, Enneper, Cayley, and others; it is composed of the surfaces all the lines of curvature of which are either plane or spherical. It may be remarked that the analysis connected with their construction is substantially simplified by the use of the well-known theorems that, if a line of curvature be either plane or spherical, the plane or the sphere on which it lies cuts the surface at an angle that remains constant along the line.

Without attempting to give a complete specification of the classes of surfaces of the kind indicated*, my object in this note is to indicate a mode of approaching the problem which appears to me to be different from the modes adopted by the authors quoted.

General equations.

The required surface is referred to its lines of curvature as the parametric curves: the parameters are taken to be p and q . The fundamental magnitudes of the first order, as they arise in Gauss's theory, are denoted by E, F, G ; those of the second order by L, M, N . Because the parametric curves are lines of curvature, we have

$$F=0, \quad M=0.$$

Let s denote the arc of a line $p = \text{constant}$, measured from a fixed point on that line; and let t denote the arc of a line $q = \text{constant}$, similarly measured from a fixed point on the line.

* A useful bibliography of the subject will be found in Loria's treatise *Il passato ed il presente delle principali teorie geometriche* (1896), pp. 162, 163.

(Both sets of fixed points may be regarded as given by director-curves.) We have

$$ds = G^{\frac{1}{2}} dq, \quad dt = E^{\frac{1}{2}} dp.$$

The circular curvature (say ρ^{-1}) and the torsion (say τ^{-1}) of a line $p = \text{constant}$ may vary from point to point upon the line, and so they may be functions of s : they may also vary from line to line, and so they may be functions of p ; hence we write

$$\rho = f(s, p), \quad \tau = h(s, p).$$

Further, at the point in question, let ψ denote the angle between the normal to the surface and the principal normal to the curve. Then

$$\frac{1}{\tau} = \frac{d\psi}{ds},$$

so that

$$\frac{d\psi}{dq} = \frac{G^{\frac{1}{2}}}{h(s, p)} \dots\dots\dots(1),$$

where

$$\frac{ds}{dq} = G^{\frac{1}{2}} \dots\dots\dots(2).$$

By the known results, giving the curvature of a normal section of the surface through the tangent to $p = \text{constant}$ and the geodesic curvature of $p = \text{constant}$, we have

$$\frac{N}{G} = \frac{\cos \psi}{f(s, p)} \dots\dots\dots(3),$$

and

$$\frac{1}{2GE^{\frac{1}{2}}} \frac{\partial G}{\partial p} = - \frac{\sin \psi}{f(s, p)} \dots\dots\dots(4).$$

Similarly, denoting by ρ'^{-1} and τ'^{-1} the circular curvature and the torsion of $q = \text{constant}$ at the point on the line determined by t , and by ψ' the angle between the normal to the surface at the point and the principal normal to the line, we have

$$\rho' = g(t, q), \quad \tau' = k(t, q),$$

$$\frac{1}{\tau'} = \frac{d\psi'}{dp},$$

so that

$$\frac{d\psi'}{dp} = \frac{E^{\frac{1}{2}}}{k(t, q)} \dots\dots\dots(5),$$

where

$$\frac{dt}{dp} = E^{\frac{1}{2}} \dots\dots\dots(6).$$

Also, as for $p = \text{constant}$, we have

$$\frac{L}{E} = \frac{\cos \psi'}{g(t, q)} \dots \dots \dots (7),$$

$$\frac{1}{2EG^{\frac{1}{2}}} \frac{\partial E}{\partial q} = - \frac{\sin \psi'}{g(t, q)} \dots \dots \dots (8).$$

But we also have to consider the intrinsic equations satisfied by all surfaces. These are: (i) Gauss's characteristic equation which, in the present case, has the form

$$4LNEG = E \left\{ \frac{\partial E}{\partial q} \frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p} \right)^2 \right\} + G \left\{ \left(\frac{\partial E}{\partial q} \right)^2 + \frac{\partial E}{\partial p} \frac{\partial G}{\partial p} \right\} \\ - 2EG \left(\frac{\partial^2 E}{\partial q^2} + \frac{\partial^2 G}{\partial p^2} \right) \dots \dots \dots (9);$$

and (ii) the two Codazzi-Mainardi equations which now are

$$\frac{\partial L}{\partial q} = \frac{1}{2} \left(\frac{L}{E} + \frac{N}{G} \right) \frac{\partial E}{\partial q} \dots \dots \dots (10),$$

$$\frac{\partial N}{\partial p} = \frac{1}{2} \left(\frac{L}{E} + \frac{N}{G} \right) \frac{\partial G}{\partial p} \dots \dots \dots (11).$$

These eleven equations constitute the aggregate for the determination of the surface. There are, initially, eight unknown quantities, viz. $s, \psi; t, \psi'; E, G; L, N$. They are expressed, in terms of p and q , by means of equations (1)–(8); if derived through those equations, the expressions will contain arbitrary elements. The equations (9), (10), (11) will impose limitations upon these arbitrary elements; and, in special cases, they may impose conditions upon $f, h, \psi; g, k, \psi'$.

Special illustration.

As an illustration of the detailed analysis required for the complete determination of the surfaces when systems of lines of curvature are given, let it be required to obtain all the surfaces which have circles for their lines of curvature.

In that case, we have

$$\rho = f(p), \quad \tau^{-1} = 0;$$

$$\rho' = g(q), \quad \tau'^{-1} = 0;$$

$$\text{hence} \quad \frac{d\psi}{dq} = 0, \quad \frac{d\psi'}{dp} = 0.$$

Accordingly, we may take

$$\psi = P, \quad \psi' = Q,$$

where P is a function of p only which may be a constant (zero included), and Q is a function of q^* only which may be a constant (zero included).

There are several cases according as P and as Q do not vanish in general or do vanish in general. They will be taken in turn.

CASE I. It is assumed that neither P nor Q vanishes in general. The equations, particular to the lines of curvature, now are

$$\frac{N}{G} = \frac{\cos P}{f(p)} \dots\dots\dots (3)',$$

$$\frac{1}{2GE^{\frac{1}{2}}} \frac{\partial G}{\partial p} = -\frac{\sin P}{f(p)} \dots\dots\dots (4)',$$

$$\frac{L}{E} = \frac{\cos Q}{g(q)} \dots\dots\dots (7)',$$

$$\frac{1}{2EG^{\frac{1}{2}}} \frac{\partial E}{\partial q} = -\frac{\sin Q}{g(q)} \dots\dots\dots (8)';$$

and there are the three equations (9), (10), (11), which must be satisfied by all surfaces.

The equations (4)' and (8)' serve to determine E and G ; and it is easy to obtain their most general primitive in the form

$$E^{-\frac{1}{2}} \frac{dH}{dp} = (H + K) \frac{\sin P}{f(p)},$$

$$G^{-\frac{1}{2}} \frac{dK}{dq} = (H + K) \frac{\sin Q}{g(q)},$$

where, so far as (4)' and (8)' are concerned, H is an arbitrary function of p , and K is an arbitrary function of q . The equations (3)' and (7)' then determine L and N ; and then all the equations, particular to the lines of curvature, are satisfied by the values of E , G , L , N which have now been obtained.

When the values are substituted in (10), we have

$$\frac{\cos P}{f(p)} - \frac{\cos Q}{g(q)} = \bar{Q}H(p) + \bar{Q}K(q) \dots\dots\dots (10)',$$

where

$$\bar{Q} \frac{dK}{dq} = -\frac{d}{dq} \left\{ \frac{\cos Q}{g(q)} \right\},$$

* The results, that ψ is constant along a line $p=\text{constant}$, and that ψ' is constant along a line $q=\text{constant}$, are Joachimsthal's theorem on plane lines of curvature.

so that $\bar{Q}$ does not involve p . The variables p and q are independent of one another: hence, in order that (10) may be satisfied, we must have

$$\bar{Q} = \text{constant} = \frac{1}{\alpha},$$

and then (10)' becomes

$$H(p) - \alpha \frac{\cos P}{f(p)} = - \left\{ K(q) + \alpha \frac{\cos Q}{g(q)} \right\}.$$

Each side of this equation must be constant, again because p and q are independent variables; denoting the common value by $-\beta$, we have

$$\left. \begin{aligned} H &= \alpha \frac{\cos P}{f(p)} - \beta \\ K &= -\alpha \frac{\cos Q}{g(q)} + \beta \end{aligned} \right\}.$$

The same results follow from substituting in (11): so that the equations (10) and (11) are satisfied in virtue of these values of H and K .

There remains equation (9). Writing

$$\begin{aligned} \frac{\cos P}{f(p)} &= P_1, & \frac{\cos Q}{g(q)} &= Q_1, \\ \frac{\sin P}{f(p)} &= P_2, & \frac{\sin Q}{g(q)} &= Q_2, \end{aligned}$$

and substituting in (9), we have

$$\begin{aligned} P_1 Q_1 + P_1 Q_2 \frac{Q_2'}{Q_1} + Q_1 P_2 \frac{P_2'}{P_1} &= \left(P_1 P_2 \frac{P_2'}{P_1} - P_2^2 \right) + \left(Q_1 Q_2 \frac{Q_2'}{Q_1} - Q_2^2 \right), \\ &= A + B \dots \dots \dots (9), \end{aligned}$$

where A is the function of p composed of the first two terms on the right-hand side, and B is the function of q composed of the last two terms. Now this relation must be satisfied identically, as p and q are independent variables; hence

$$\begin{aligned} P_1' \left(Q_1 + Q_2 \frac{Q_2'}{Q_1} \right) + Q_1 \frac{d}{dp} \left(P_2 \frac{P_2'}{P_1} \right) &= A', \\ P_1 \frac{d}{dq} \left(Q_1 + Q_2 \frac{Q_2'}{Q_1} \right) + Q_1' P_2 \frac{P_2'}{P_1} &= B', \\ P_1' \frac{d}{dq} \left(Q_1 + Q_2 \frac{Q_2'}{Q_1} \right) + Q_1' \frac{d}{dp} \left(P_2 \frac{P_2'}{P_1} \right) &= 0 \end{aligned}$$

From the last of these derived relations, we have

$$\frac{d}{dq} \left(Q_1 + Q_2 \frac{Q_2'}{Q_1'} \right) = a' Q_1',$$

$$\frac{d}{dp} \left(\frac{P_2 P_2'}{P_1'} \right) = -a' P_1',$$

where a' is a constant; and therefore

$$Q_1 + Q_2 \frac{Q_2'}{Q_1'} = a' Q_1 + b',$$

$$P_2 \frac{P_2'}{P_1'} = -a' P_1 + c',$$

where b' and c' are constants. Hence

$$Q_1^2 + Q_2^2 = a' Q_1^2 + 2b' Q_1 + m,$$

$$P_2^2 = -a' P_1^2 + 2c' P_1 + m',$$

where m and m' are constants. But it is equation (9)', and not merely any derived relation, which has to be satisfied; when we substitute, we find

$$b' P_1 + c' Q_1 = -c' P_1 - b' Q_1 - (m + m'),$$

and therefore

$$b' = -c', \quad m + m' = 0.$$

Accordingly, all the equations are satisfied by the relations

$$\left. \begin{aligned} Q_1^2 + Q_2^2 &= a' Q_1^2 - 2c' Q_1 + m \\ P_2^2 &= -a' P_1^2 + 2c' P_1 - m \\ H &= \alpha P_1 - \beta \\ K &= -\alpha Q_1 + \beta \end{aligned} \right\},$$

where a' , c' , m are arbitrary constants, as also are α and β : but when the values of E , G , L , N are formed, α and β disappear. Thus there are three arbitrary constants at our disposal.

The results can be made simpler in form by appropriate changes in the independent variables. Let a new variable ϕ be defined by the relation

$$\frac{1}{P_1} = \mu - \alpha \cos \phi,$$

where μ and a are constants: then

$$\frac{P_2}{P_1} = \frac{a}{ib} \sin \phi,$$

provided

$$m = -\frac{1}{b^2}, \quad c' = m\mu, \quad a' = m(\mu^2 - a^2).$$

Also, let another new variable θ be defined by the relation

$$\frac{1}{Q_1} = \mu' - c \cos \theta;$$

then

$$\frac{Q_2}{Q_1} = \frac{c}{b} \sin \theta,$$

provided

$$m = -\frac{1}{b^2}, \quad c' = m\mu', \quad a' = 1 + m(\mu'^2 - c^2).$$

All these relations among the constants can be simultaneously satisfied if

$$\mu = \mu', \quad c^2 = a^2 - b^2;$$

and then, in place of the three constants a', c', m , we have the four constants a, b, c, μ , subjects to the relation

$$c^2 = a^2 - b^2.$$

Accordingly, the results can be expressed in the form

$$\left. \begin{aligned} \frac{\cos P}{f(p)} &= P_1 = \frac{1}{\mu - a \cos \phi} \\ \tan P &= \frac{P_2}{P_1} = \frac{a}{ib} \sin \phi \\ \frac{\cos Q}{g(q)} &= Q_1 = \frac{1}{\mu - c \cos \theta} \\ \tan Q &= \frac{Q_2}{Q_1} = \frac{c}{b} \sin \theta \end{aligned} \right\}.$$

Let $E', 0, G', L', 0, N'$ denote the new expressions for $E, 0, G, L, 0, N$, when taken in terms of the new variables; then, as

$$E = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2, \quad E' = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2,$$

we have
$$E = E' \left(\frac{\partial \phi}{\partial p} \right)^2;$$

and similarly
$$G = G' \left(\frac{\partial \theta}{\partial q} \right)^2.$$

Hence
$$\frac{1}{E'^{\frac{1}{2}}} \frac{dH}{d\phi} = \{H(\phi) + K(\theta)\} P_2,$$

so that
$$E' = - \frac{b^2 (\mu - c \cos \theta)^2}{(a \cos \phi - c \cos \theta)^2}.$$

Similarly
$$G' = \frac{b^2 (\mu - a \cos \phi)^2}{(a \cos \phi - c \cos \theta)^2}.$$

The values of L' and N' are given by

$$\frac{L'}{E'} = \frac{L}{E} = Q_1 = \frac{1}{\mu - c \cos \theta},$$

$$\frac{N'}{G'} = \frac{N}{G} = P_1 = \frac{1}{\mu - a \cos \phi}.$$

Thus the fundamental magnitudes of the first order and of the second order are explicitly expressed in terms of two independent variables which, in the present case, give families of lines of curvature by their parametric values.

It is known, from a theorem established* by Bonnet, that a surface is uniquely determined (save as to orientation and position, which do not affect its intrinsic properties), by the assignment of the values of these magnitudes, if only they satisfy the equations which are characteristic of all surfaces. Consequently, it is sufficient to show that the necessary conditions, with the necessary generality, are satisfied by a particular class of surfaces: and so it will be proved that, within this range, all the surfaces in question are Dupin's cyclides, by identifying them with the general relations.

The formal results as regards Dupin's cyclides are easily obtained by using the property of the surface that it is the envelope of a sphere, which has its centre on one conic and passes through one particular point on another conic, provided that the two conics are confocal and that they lie† in perpendicular planes. Also the generation is double: that is, either of the conics can be taken as the locus of the centre of the moving sphere, subject to a relation concerning the fixed points of passage on the respective conics.

* *Journal de l'École Polytechnique*, cah. XLII. (1867), p. 31.

† Cayley, *Collected Mathematical Papers*, vol. IX., p. 65.

A brief statement of the analysis will be sufficient. Let the two conics be given by the equations

$$\left. \begin{aligned} \frac{x^2}{a'^2} + \frac{y^2}{b'^2} &= 1 \\ z &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{x^2}{c'^2} - \frac{z^2}{b'^2} &= 1 \\ y &= 0 \end{aligned} \right\};$$

the condition that the conics are confocal requires the relation

$$c'^2 = a'^2 - b'^2.$$

Let the centre of the sphere, which moves on the first conic, be denoted by $a' \cos \theta'$, $b' \sin \theta'$, 0; and let the point on the second conic, through which all the moving spheres pass, be α , 0, γ . The equation of the sphere is

$$x^2 + y^2 + z^2 - \alpha^2 - \gamma^2 - 2(x - \alpha)a' \cos \theta' - 2b'y \sin \theta' = 0.$$

The envelope of this sphere is

$$(x^2 + y^2 + z^2 - \alpha^2 - \gamma^2)^2 = 4a'^2(x - \alpha)^2 + 4b'^2y^2;$$

and the circular lines of curvatures, given by the intersection of successive spheres, are represented by the equations

$$\left. \begin{aligned} x^2 + y^2 + z^2 - \alpha^2 - \gamma^2 - 2(x - \alpha)a' \cos \theta' - 2b'y \sin \theta' &= 0 \\ (x - \alpha)a' \sin \theta' - b'y \cos \theta' &= 0 \end{aligned} \right\}.$$

For the other generation, let the centre of the sphere be at the point $c' \cos \phi'$, 0, $ib' \sin \phi'$, and suppose that the point common to all the spheres is α' , β' , 0. The equation of the sphere is

$$x^2 + y^2 + z^2 - \alpha'^2 - \beta'^2 - 2(x - \alpha')c' \cos \phi' - 2ib'z \sin \phi' = 0,$$

and the envelope of this sphere is

$$(x^2 + y^2 + z^2 - \alpha'^2 - \beta'^2)^2 = 4c'^2(x - \alpha')^2 + 4b'^2z^2;$$

and the corresponding circular lines of curvature, given by the intersection of successive spheres, are represented by the equations

$$\left. \begin{aligned} x^2 + y^2 + z^2 - \alpha'^2 - \beta'^2 - 2(x - \alpha')c' \cos \phi' - 2ib'z \sin \phi' &= 0 \\ (x - \alpha')c' \sin \phi' - ib'z \cos \phi' &= 0 \end{aligned} \right\}.$$

Having regard to the relations

$$\frac{\alpha^2}{c'^2} - \frac{\lambda^2}{b'^2} = 1, \quad \frac{a'^2}{\alpha^2} + \frac{\beta'^2}{b'^2} = 1, \quad c'^2 = a'^2 - b'^2,$$

it is easy to prove that the two envelopes of the two sets of moving spheres are the same surface if

$$a'^2\alpha = c'^2\alpha' = a'c'\mu,$$

where μ is the parameter determining the simultaneous values of α and α' . With these values, we have

$$\left. \begin{aligned} x &= \mu \frac{a'}{c'} + \frac{b'^2}{c'} \frac{(c' \cos \theta' - \mu) \cos \phi'}{a' \cos \phi' - c' \cos \theta'} \\ y &= \frac{b' (a' \cos \phi' - \mu)}{a' \cos \phi' - c' \cos \theta'} \sin \theta' \\ z &= i \frac{b' (c' \cos \theta' - \mu)}{a' \cos \phi' - c' \cos \theta'} \sin \phi' \end{aligned} \right\}$$

as the expressions for the coordinates of any point on the surface.

When the fundamental magnitudes which determine the surface intrinsically are constructed, we find

$$E' = \left(\frac{\partial x}{\partial \phi'} \right)^2 + \left(\frac{\partial y}{\partial \phi'} \right)^2 + \left(\frac{\partial z}{\partial \phi'} \right)^2 = - \frac{b'^2 (\mu - c' \cos \theta')^2}{(a' \cos \phi' - c' \cos \theta')^2},$$

$$F' = \frac{\partial x}{\partial \theta'} \frac{\partial x}{\partial \phi'} + \frac{\partial y}{\partial \theta'} \frac{\partial y}{\partial \phi'} + \frac{\partial z}{\partial \theta'} \frac{\partial z}{\partial \phi'} = 0,$$

$$G' = \left(\frac{\partial x}{\partial \theta'} \right)^2 + \left(\frac{\partial y}{\partial \theta'} \right)^2 + \left(\frac{\partial z}{\partial \theta'} \right)^2 = \frac{b'^2 (\mu - a' \cos \phi')^2}{(a' \cos \phi' - c' \cos \theta')^2},$$

$$L' = - \frac{b'^2 (\mu - c' \cos \theta')}{(a' \cos \phi' - c' \cos \theta')},$$

$$M = 0,$$

$$N' = \frac{b'^2 (\mu - a' \cos \phi')}{(a' \cos \phi' - c' \cos \theta')^2},$$

The fundamental magnitudes, for the general surface and for the cyclide in particular, agree if

$$a = a', \quad c = c', \quad \theta = \theta', \quad \phi = \phi', \quad b = b';$$

that is, all the surfaces are Dupin cyclides.

CASE II. Now let it be assumed that one of the two quantities P and Q vanishes, but not both of them; take

$$P = 0.$$

Then, by (4)', G is a function of q only; we can effectively take G equal to unity by introducing a new variable q' such that $dq'^2 = G dq^2$; hence no generality is lost by assuming

$$G = 1.$$

Also, by (3)', we have

$$N = \frac{1}{f(p)}.$$

Inserting the value of G in (11), we have

$$\frac{\partial N}{\partial p} = 0,$$

so that N is a pure constant. As $G \div N$ is thus a constant length, we take it to be the actual unit of length; so

$$N = 1.$$

Again, the equation (8)' gives a value of E in a form

$$E = \bar{P} e^{-2 \int \{\sin Q\} / \{g(q)\} dq},$$

where $\bar{P}$ is a function of p alone; the actual value of E is not, at the moment, of immediate importance. Further, by (7)', we then can derive the value of L , not in that form of any immediate importance; but using the equation

$$\frac{L}{E} = Q_1,$$

taking logarithmic derivatives, and having regard to (10) and to (9), we find

$$Q_1' = Q_2 (Q_1 - 1),$$

$$Q_2' = Q_2^2 + Q_1,$$

the quantities Q_1 and Q_2 being functions of q alone. The primitive of these equations is

$$\left. \begin{aligned} Q_1 - 1 &= - \frac{c}{c + \cos q} \\ Q_2 &= \frac{\sin q}{c + \cos q} \end{aligned} \right\},$$

where c is an arbitrary constant, and where q has absorbed an additive arbitrary constant. Hence

$$\left. \begin{aligned} P &= 0 \\ f(p) &= 1 \end{aligned} \right\}, \quad \left. \begin{aligned} Q &= q \\ g(q) &= c + \cos q \end{aligned} \right\};$$

and therefore

$$\left. \begin{aligned} E &= (c + \cos q)^2, & F &= 0, & G &= 1 \\ L &= c + \cos q, & M &= 0, & N &= 1 \end{aligned} \right\}.$$

These are the intrinsic equations of an *anchor-ring*.

CASE III. Lastly, let it be assumed that both P and Q vanish. The case is relatively unimportant in detail: for by (4)', we have G a function of q alone: and, by (8)', E is a function of p alone: thus, by (10), we have L a function of p alone; and, by (11), we have N a function of q alone. Moreover, by (9), we then have

$$LNEG = 0,$$

which shews that the surface either is a *developable surface* or is *deformable into a developable surface*. The case demands no further consideration.

March, 1903.

A NOTE ON TRIGONOMETRICAL SERIES.

By *W. H. Young, Sc.D., F.R.S.*

§1. THE fundamental theorem in the theory of trigonometrical series is that which states that two distinct trigonometrical series cannot exist which converge to the same value for all points of the interval $(-\pi, \pi)$ with the exception of a reducible set of points at which the series are not known to converge to the same value, or to converge at all. It is of interest to notice that the well-known proof of this theorem, which is explained in §§486, 487 of Hobson's "Treatise on the Theory of Functions of a Real Variable," suffices, with a slight modification, to prove a much more general theorem, viz. that it is sufficient for the set to be *countable*, so that it need not be reducible and may be everywhere dense.

In other words, *if a trigonometrical series converges to the value zero at all points of an interval except, possibly, at a countable set of points, it necessarily converges everywhere to zero.*

§2. The modification in question depends on a similar and more general formulation of Harnack's theorem that if the points at which the measure of oscillation is $\geq k$ form a non-dense set, then

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

where the series is supposed written in the form

$$\frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx),$$

The new formulation is as follows:—

THEOREM. *If the points of divergence of a trigonometrical series form a set of the first category*, then*

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

For in this case the points at which the oscillation $\geq k$ could not be everywhere dense in any interval, since by Theorem 3, Cor. 2 of my paper "On Oscillating Successions of Continuous Functions†" the points at which the oscillation is $> k$ form an inner limiting set and therefore in any interval in which they are everywhere dense form a set of the second category, contrary to hypothesis. Thus the points at which the oscillation $> k$ cannot, for any value of k , be everywhere dense, whence neither can the points of oscillation $\geq k$ be everywhere dense in any interval, and therefore form a set dense nowhere. Hence, by Harnack's theorem, stated in Hobson's book § 486, the result follows.

§ 3. We have now merely to omit the words "reducible and therefore" in line 3 § 487 and change the word "reducible" in line 14 and in the final enunciation on p. 749 of Hobson's book, and the proof given in that article applies, as it stands, to "countable."

The result thus obtained has obviously a much greater generality than the classical theorem. We may state it as follows:—

THEOREM. *If the values of a function be assigned at all but a countable set of points, it can be expressed as a trigonometrical series in at most one way.*

§ 4. The interesting question naturally presents itself as to whether this result is capable of still further extension. As far as I know no proof of a more general result has yet been

* See Young's "Theory of Sets of Points," Camb. Univ. Press, 1906, pp. 70, 71. A set of the first category was defined by Baire as a set which can be generated as the outer limiting set of sets, not necessarily closed, which are nowhere dense. A set of the second category is one which cannot be so generated. A special case of a set of the second category is the complementary set of a set of the first category. I repeat these definitions explicitly because considerable confusion has been occasioned by the original mistake in Schoenflies's Bericht, and the consequent lack of precision in the use of the term "set of the second category" among subsequent writers. Lebesgue has shown that the continuum may be divided into two sets of the second category at least theoretically.

† *Proc. L.M.S.*, Series 2, vol. VI., p. 303.

‡ The theorems in this and the preceding articles were among others communicated by me without proof to the 4th Mathematical Congress at Rome in April of the present year and some weeks previously by letter to Dr. Hobson.

published. Professor Bernstein, however, informs me that he is about to publish one. His theorem, which, he tells me, was communicated some months ago to the Göttinger Mathematische Gesellschaft, gives the extreme limits of possibility in question, viz. the set of doubtful points may be any which does not contain any perfect component.

Such sets are theoretically more general than countable sets, though it would appear that they are, from their very nature, incapable of being defined by any simple law, so that they are not likely to arise except in the form of countable sets. Professor Bernstein tells me, however, that he has demonstrated their existence as theoretical sets of points. I terminate this note by showing that these sets of Bernstein's which, of course, are of zero inner content, belong to the first category, or, if not, are at any rate in every interval complementary to a set of the second category. This follows as a corollary to the following theorem.

§4. THEOREM. *Let G be a set of the first category and A any point not belonging to it. Then it is possible to find a perfect set containing A and having no point common with the set of the first category G .*

Let the sets, dense nowhere, which define G be $G_1, G_2, \dots$, each of these containing its predecessor and other points beside. Since G_1 is dense nowhere, there is in every interval with A as end-point an interval entirely free of points of G_1 . Hence we can find a set D_0 of non-overlapping and non-abutting intervals having A as sole and only limiting point and having no point common with G_1 . Divide each of these intervals into 5 equal parts and omit the two extreme and the central part. We thus get a set of intervals D_1 with the same properties as D_0 .

Inside each of the intervals of D_1 we can find an interval free of points of G_2 , since G_2 is dense nowhere. We divide this interval as before into 5 equal parts and take the second and fourth parts. We thus get a set of non-overlapping and non-abutting intervals D_2 , internal to D_1 , having A as limiting point and having no point common with G_2 .

Continuing this process we get a series of sets of intervals $D_1, D_2, \dots, D_n, \dots$ each lying inside the preceding. Moreover, since in each interval of D_n there is a pair of intervals of D_{n+1} , it follows* that the inner limiting set E is of potency c . Each set of intervals D_n has A as limiting point, therefore A

* See my paper in the *Leipziger Bericht*, Aug., 1903.

is a limiting point of E . This point A is the only point needed to close E , since it is the only point needed to close each set of intervals, so that A together with E is the deduced set of the closed sets D_n together with E , and is therefore a closed set by Cantor's Theorem of Deduction. Moreover, the mode of proof used in proving that the set E is of potency c shows also that there are c points of E in every interval containing A , so that A is a point of the perfect nucleus N of the closed set E with A . The nucleus N is therefore a perfect set containing A and having no point common with the set of the first category, which proves the theorem.

Cor. 1. A set which has a point other than A common with every perfect set, containing a given point A , is a set of the second category.

Cor. 2. A set which has no perfect component is complementary to a set of the second category.)

For, if N is any perfect set, it is not a component of the set which has no perfect component, and has therefore a point common with the complementary set. This complementary set, being such that there is no perfect set with which it has not a common point, is not, therefore, a set of the first category, by the above theorem. By Baire's definition, therefore, it is a set of the second category.

Cor. 3. The complementary set of a set of the first category has a perfect component. (This is a particular case of a set of the second category.)

Cor. 4. Every point of a set whose complementary set is of the first category is of degree c in the set. Such a set is therefore its own ultimate coherence (total inheritance).*

§5. The preceding theorem may be generalised, taking instead of a single point A , any closed set. The proof depends on the following simple Lemma:—

Lemma. If $A_1, A_2, \dots$ is a countable set of points, such that for each index i the point A_i belongs to a closed set E_i of span† $\leq k_i$, ($k_1, k_2, \dots$ being a sequence with zero as limit),

* For the meaning of this word in this connection see Young's "Theory of Sets of Points," p. 53.

† For the meaning of this word see Young's "Theory of Sets of Points," p. 187. Q.E.D.

then the set E consisting of all the sets E_i together with the limiting points $A_1, A_2, \dots$ is a closed set.

For let P be any point not belonging to E . Certainly, therefore P is not one of the points A_i , nor one of their limiting points. Therefore we can find a region with P as internal point containing none of the points A_i . Let the minimum distance of P from the boundary of this region be s . Then since the quantities k_i form a sequence with zero as limit, we can determine m so that, for all values of $n \leq m$,

$$k_n < \frac{1}{2}s$$

It then follows that all the sets E_n , for $n \geq m$, are at a minimum distance $> \frac{1}{2}s$ from P , so that the only points of E within a distance $\frac{1}{2}s$ of P belong to $E_1, E_2, \dots, E_m$. But P is not a point of $E_1, E_2, \dots$ or E_m , and therefore has a positive minimum distance from each. If d is the smallest of these minimum distances, it follows that there are no points of E within a distance d of P . Thus P is not a limiting point of E , so that E is a closed set.

The extended theorem is now as follows:—

THEOREM 2. *If A is any closed set of points, having no point common with a set G of the first category, it is possible to find a perfect set containing the set A and having no point common with G .*

For let $A_1, A_2, \dots$ be a countable number of points dense everywhere in the closed set A . By theorem 1 we can, for each value of the index i , take an interval of length 2^{-i} having A_i as middle point, and find in this interval a perfect set N_i containing A_i and having no point common with G .

These sets N_i and the points A_i then satisfy the conditions of the Lemma, so that, A being the set consisting of the points A_i and their limiting points, the set E which consists of the set A and all the perfect sets N_i is a closed set. It is, moreover, dense in itself, since each of its points is a limiting point of the points A_i , or is a point of one of the perfect components N_i . Thus the set E is perfect, contains the set A , and obviously has no point common with G , which proves the theorem.

A FORMULA IN FINITE DIFFERENCES AND ITS APPLICATION TO MECHANICAL QUADRATURE.

By *S. T. Shovelton*, Merton College, Oxford.

1. IF u_x can be expanded in a series of ascending powers of x , then $u_x + u_{-x}$ can be expanded in a series of ascending even powers of x .

For shortness write $2f(x) = u_x + u_{-x}$ and assume

$$f(x) = a + bx^2 + cx^2(x^2 - h^2) + dx^2(x^2 - h^2)(x^2 - 2^2h^2).$$

(This assumption implies that seventh differences of u_x are constant).

In order to find the coefficients a, b, c , and d it is necessary to know $f(x)$ for four different values of x .

If the known values are $f(0), f(h), f(2h), f(3h)$, we have

$$f(0) = a,$$

$$f(h) = a + bh^2,$$

$$f(2h) = a + 4bh^2 + 12ch^4,$$

$$f(3h) = a + 9bh^2 + 72ch^4 + 360dh^6;$$

therefore

$$a = f(0) = u_0,$$

$$bh^2 = f(h) - f(0) = \frac{1}{2}(u_h - 2u_0 + u_{-h}) = \frac{1}{2}\Delta^2 u_{-h},$$

$$12ch^4 = f(2h) - 4f(h) + 3f(0) = \frac{1}{2}\Delta^4 u_{-2h},$$

$$360dh^6 = f(3h) - 6f(2h) + 15f(h) - 10f(0) = \frac{1}{2}\Delta^6 u_{-3h},$$

where the Δ refers to differences of h in the values of x .

Therefore

$$f(x) = \left[u_0 + \frac{x^2}{2!h^2} \Delta^2 u_{-h} + \frac{x^2(x^2 - h^2)}{4!h^4} \Delta^4 u_{-2h} + \frac{x^2(x^2 - h^2)(x^2 - 2^2h^2)}{6!h^6} \Delta^6 u_{-3h} \right]$$

on the assumption that seventh differences of u_x are constant.

2. The form of $f(x)$ suggests the probability that for any rational integral algebraic function of x of finite order,

$$f(x) = u_0 + \sum_{r=1} \frac{x^2(x^2-h^2)\dots\{x^2-(r-1)^2h^2\}}{(2r)!h^{2r}} \Delta^{2r} u_{-rh}.$$

For shortness denote

$$\frac{x^2(x^2-h^2)(x^2-2^2h^2)\dots\{x^2-(r-1)^2h^2\}}{(2r)!h^{2r}}$$

by $[x^2]^{(r)}$.

Then it can readily be proved that $\Delta^2[x^2]^{(r)} = [(x+h)^2]^{(r-1)}$.

$$\text{Assume } f(x) = a_0 + a_2[x^2]^{(1)} + a_4[x^2]^{(2)} + \dots \dots \dots (1).$$

If u_x is a rational integral function of x of finite order this series will terminate. Under this supposition, we have

$$\Delta^2 f(x) = a_2 + a_4[(x+h)^2] + a_6[(x+h)^2]^{(2)} + \dots \dots (2),$$

$$\Delta^4 f(x) = a_4 + a_6[(x+2h)^2] + \dots \dots \dots (3),$$

$$\Delta^{2r} f(x) = a_{2r} + a_{2r+2}[(x+rh)^2] + \dots \dots \dots (r+1).$$

Putting $x=0, -h, -2h$, etc, in the equations (1), (2), (3), etc. respectively, we have

$$a_0 = f(x)$$

$$a_2 = [\Delta^2 f(x)]_{x=-h},$$

$$a_4 = [\Delta^4 f(x)]_{x=-2h},$$

$$\text{and in general } a_{2r} = [\Delta^{2r} f(x)]_{x=-rh}.$$

$$\text{Now } \Delta^{2r} u_{-x} = u_{-(x+2rh)} - {}^{2r}C_1 u_{-(x+(2r-1)h)} + \dots + u_{-x},$$

$$\begin{aligned} \text{therefore } (\Delta^{2r} u_{-x})_{x=-rh} &= u_{-hr} - {}^{2r}C_1 u_{-(r-1)h} + \dots + u_{hr} \\ &= \Delta^{2r} u_{-hr}, \end{aligned}$$

$$\text{therefore } [\Delta^{2r} f(x)]_{x=-hr} = \Delta^{2r} u_{-hr},$$

$$\text{therefore } f(x) = u_0 + \sum_{r=1}^{r=m} [x^2]^{(r)} \Delta^{2r} u_{-hr} \dots \dots \dots (A),$$

where u_x is a rational integral algebraic function of x of degree $2m$ or $(2m+1)$.

3. Further, it can readily be verified that equation (A) holds for all values of x of the form hy where y is any integer from $-m$ to $+m$ inclusive, no matter what the form of u_x is, provided it be finite.

The general case.

4. If u_x is any function of x which is finite and continuous from $x = -mh$ to $x = +mh$ inclusive, then

$$f(x) = u_0 + \sum_{r=1}^{r=m} [x^2]^{(r)} \Delta^{2r} u_{-hr} + \frac{x^2 - m^2 h^2}{x} [x^2]^{(m)} \left[\Delta^{2m} \frac{f(x) - f(y)}{x - y} \right]_{y=-mh},$$

where Δ in the last term operates only on y .

$$\text{Let } \phi(z) = f(x) - u_0 - \sum_{r=1}^{r=m} [z^2]^{(r)} \Delta^{2r} u_{-hr}.$$

Then $\phi(z)$ is a function of z of degree $2m$.

We have, from § 3, $\phi(0) = f(x) - f(0)$,

$$\phi(\pm h) = f(x) - f(\pm h),$$

and in general $\phi(\pm rh) = f(x) - f(\pm rh)$.

We thus know the value of $\phi(z)$ for $(2m+1)$ values of z , and therefore, by Lagrange's interpolation formula,

$$\begin{aligned} \phi(z) &= \frac{(z+mh)\{z+(m-1)h\} \dots \{z-h\} \dots \{z-mh\}}{(2m)! h^{2m}} \\ &\times \left[\frac{\phi(-mh)}{z+mh} {}_{-2m}C_1 \cdot \frac{\phi\{-(m-1)h\}}{z+(m-1)h} + \dots - {}_{-2m}C_1 \frac{\phi\{(m-1)h\}}{z-(m-1)h} + \frac{\phi(mh)}{z-mh} \right] \\ &= \frac{z(z^2-h^2)(z^2-2^2h^2) \dots (z^2-m^2h^2)}{(2m)! h^{2m}} \left[\Delta^{2m} \frac{\phi(y)}{z-y} \right]_{y=-mh} \dots (B), \end{aligned}$$

where the Δ refers as usual to differences of h and operates only on y .

Therefore

$$f(x) = u_0 + \sum_{r=1}^{r=m} [z^2]^{(r)} \Delta^{2r} u_{-hr} + \frac{z(z^2-h^2) \dots (z^2-m^2h^2)}{(2m)! h^{2m}} \left[\Delta^{2m} \frac{f(x) - f(y)}{z-y} \right]_{y=-mh}.$$

This is an equation in z of the $(2m)^{\text{th}}$ degree which is satisfied for $(2m+1)$ values of z . It is therefore an identity.

Putting $z = x$, we have

$$f(x) = u_0 + \sum_{r=1}^{r=m} [x^2]^{(r)} \Delta^{2r} u_{-hr} \\ + \frac{x^2 - m^2 h^2}{x} \cdot [x^2]^{(m)} \left[\Delta^{2m} \frac{f(x) - f(y)}{x - y} \right]_{y=-mh} \dots\dots (C),$$

where in the last term Δ operates on y only.

5. Since

$$\frac{f(x) - f(y)}{x - y} = \int_0^1 f' \{y + (x - y) z\} dz = f' \{y + (x - y) \theta\},$$

where θ is some proper fraction, the third term in (C) may be exhibited in the form

$$\frac{x^2 - m^2 h^2}{x} [x^2]^{(m)} [\Delta^{2m} f' \{y + (x - y) \theta\}]_{y=-mh}^*.$$

The remainder may be put into a form not involving differences, which for some forms of $f(x)$ is more useful.

Consider†

$$\phi(z) \equiv f(z) - u_0 - \sum_{r=1}^{r=m} [z^2]^{(r)} \Delta^{2r} u_{-hr} \\ - \frac{z(z^2 - h^2) \dots (z^2 - m^2 h^2)}{(2m)! h^{2m}} \left[\Delta^{2m} \frac{f(z) - f(y)}{z - y} \right]_{y=-mh}.$$

Then $\phi(z)$ vanishes when z has any of the $(2m+1)$ values

$$-mh, -(m-1)h, \dots, 0, h, \dots, mh,$$

and also for the value $z = x$.

Therefore, if x lies between $-mh$ and $+mh$, the $(2m+1)^{\text{th}}$ differential coefficient of $\phi(z)$ vanishes for some values of z between $-mh$ and $+mh$, provided that $f(z)$ and all its differential coefficients up to and including the $(2m+1)^{\text{th}}$ are finite and continuous between these limits.

* Boole, *Calculus of Finite Differences*, p. 147.

† See a paper by Mr. W. F. Sheppard in the *Proc. L.M.S.*, Series 2, vol IV., Part 5.

Therefore, if ξ be this value,

$$\left[\frac{d^{2m+1}}{dz^{2m+1}} f(z) \right]_{z=\xi} = \frac{(2m+1)}{h^{2m}} \left[\frac{\Delta^{2m} f(x) - f(y)}{x-y} \right]_{y=-mh}.$$

Therefore

$$f(x) = u_0 + \sum_{r=1}^{r=m} [x^2]^{(r)} \Delta^{2r} u_{-hr} + \frac{x(x^2 - h^2) \dots (x^2 - m^2 h^2)}{(2m+1)!} f^{(2m+1)}(\xi) \dots \dots (D),$$

where $-mh < \xi < mh$.

6. This formula is considerably more useful than the ordinary expansion of u_x in terms of $\Delta^r u_0$ when applied to the approximate evaluation of integrals between given limits.

We have

$$\begin{aligned} \int_{-mh}^{+mh} u_x dx &= \int_{-mh}^{+mh} u_{-x} dx = \int_{-mh}^{+mh} f(x) dx = 2 \int_0^{mh} f(x) dx \\ &= 2h \left[mu_0 + \frac{\Delta^2 u_{-h}}{2!} \cdot \frac{m^3}{3} + \frac{\Delta^4 u_{-2h}}{4!} \left(\frac{m^5}{5} - \frac{m^3}{3} \right) \right. \\ &+ \frac{\Delta^6 u_{-3h}}{6!} \left(\frac{m^7}{7} - 5 \frac{m^5}{5} + 4 \frac{m^3}{3} \right) + \dots \\ &+ \frac{\Delta^{2m} u_{-mh}}{(2m)! h^{2m+1}} \int_0^{mh} x^2 (x^2 - h^2) (x^2 - 2^2 h^2) \dots \{x^2 - (m-1)^2 h^2\} dx \\ &+ \frac{1}{2h} \int_{-mh}^{mh} R_x dx \Big] \dots \dots \dots (E), \end{aligned}$$

where
$$R_x = \frac{x(x^2 - h^2) \dots (x^2 - m^2 h^2)}{(2m+1)!} f^{(2m+1)}(\xi).$$

Since
$$\Delta^{2r} u_{-hr} = y_{hr} - {}^{2r}C_1 y_{(r-1)h} + \dots + (-1)^r {}^{2r}C_r y_0,$$

this expression gives the value of $\int_{-mh}^{+mh} u_x dx$ in terms of the values of u_x for $x=0, \pm h, \pm 2h, \dots, \pm mh$, the magnitude of the error being $\int_{-mh}^{mh} R_x dx$.

7. Let us consider the special case of $m=3$ and let it be required to find the value of $\int_{-3h}^{+3h} u_x dx$ given the seven values of u_x for $x=0, \pm h, \pm 2h, \pm 3h$. Then, from (E),

$$\int_{-3h}^{+3h} u_x dx = 2h \left[3u_0 + 9 \frac{\Delta^2 u_{-h}}{2!} + \frac{9 \times 22 \Delta^2 u_{-2h}}{5 \times 4!} + \frac{2 \times 32}{7} \cdot \frac{\Delta^6 u_{-3h}}{6!} \right],$$

neglecting the remainder, or, in other words, considering seventh differences of u_x constant,

$$\text{But } u = f(0),$$

$$\Delta^2 u_{-h} = 2f(h) - 2f(0),$$

$$\Delta^4 u_{-2h} = 2f(2h) - 8f(h) + 6f(0),$$

$$\Delta^6 u_{-3h} = 2f(3h) - 12f(2h) + 30f(h) - 20f(0),$$

therefore

$$\begin{aligned} \int_{-3h}^{+3h} u_x dx &= 6h [f(0) - 3f(0) + 3f(h) + \frac{3}{10}f(0) - \frac{4}{10}f(h) \\ &+ \frac{1}{10}f(2h) - \frac{4}{2}f(0) + \frac{4}{2} \cdot \frac{1}{10}f(h) - \frac{4}{7}f(2h) + \frac{4}{2}f(3h)] \dots (F) \\ &= \frac{h}{140} [272 (u_0) + 27 (u_h + u_{-h}) + 216 (u_{2h} + u_{-2h}) + 41 (u_{3h} + u_{-3h})]. \end{aligned}$$

If we multiply the last four terms of (F) by $\frac{4}{41}$, we obtain

$$\int_{-3h}^{+3h} u_x dx = \frac{3h}{10} [6u_0 + u_h + u_{-h} + 5(u_{2h} + u_{-2h}) + u_{3h} + u_{-3h}]^*.$$

The error in multiplying the last four terms of (F) by $\frac{4}{41}$ is

$$+ \frac{3h}{420} [2f(3h) - 12f(2h) + 30f(h) - 20f(0)] = + \frac{h}{140} (\Delta^6 u_{-3h}).$$

If we multiply the last four terms of (F) by $\frac{4}{41} \cdot \frac{1}{15}$ the term $f(h)$ disappears and we obtain

$$\begin{aligned} \int_{-3h}^{+3h} u_x dx &= 6h \left[\frac{1}{30}f(0) + \frac{2}{5}f(2h) + \frac{1}{15}f(3h) \right] \\ &= h [2.2u_0 + 1.62 (u_{2h} + u_{-2h}) + .28 (u_{3h} + u_{-3h})] \dots (G). \end{aligned}$$

This formula has been obtained by the use of Maclaurin's theorem by Mr. G. F. Hardy.*

The error introduced by multiplying the last four terms of (F) by $\frac{1}{4!} \cdot \frac{1}{15}$ is

$$-\frac{9h}{360} [f(3h) - 6f(2h) + 15f(h) - 10f(0)] = -\frac{9h}{700} (\Delta^6 u_{-3h}).$$

It will readily be seen from the form of the remainder

$$\int_{-3h}^{+3h} \frac{(x^2 - h^2)(x^2 - 2^2 h^2)(x^2 - 3^2 h^2)}{7!} f^{(7)}(\xi) dx,$$

that these approximations to the values of the integrals can only be safely employed when $f^{(7)}(x)$ is zero within the limits of accuracy required, that is to say, when the seventh differences of u_x are approximately constant.

8. To obtain an approximate expression for the value of $\int_0^{2mh} u_x dx$ given the value of u_x for $x = -h, h, 3h, 5h, \dots$, and given that u_x is zero for all values of x after some finite value, and that all differences above the second may be neglected.

$$\begin{aligned} \int_0^{mh} u_x dx &= \int_0^h u_x dx + \int_h^{2h} u_x dx + \dots + \int_{(m-1)h}^{mh} u_x dx \\ \int_{(r-1)h}^{rh} u_x dx &= \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} u_{(r-\frac{1}{2})h+x} dx \\ &= 2h \left[\frac{1}{2} u_{(r-\frac{1}{2})h} + \frac{1}{24} \cdot \frac{\Delta^2 u_{(r-\frac{3}{2})h}}{2!} - \frac{1}{480} \frac{\Delta^4 u_{(r-\frac{5}{2})h}}{4!} \dots \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{mh} u_x dx &= 2h \left[\frac{1}{2} \sum_{r=1}^{r=m} u_{(r-\frac{1}{2})h} + \frac{1}{24} \frac{\{\Delta u_{(m-\frac{1}{2})h} - \Delta u_{-\frac{1}{2}h}\}}{2!} \right. \\ &\quad \left. - \frac{1}{480} \frac{\{\Delta^3 u_{(m-\frac{3}{2})h} - \Delta^3 u_{-\frac{3}{2}h}\}}{4!} \dots \right] \end{aligned}$$

* *Journal of Institute of Actuaries*, vol. XXIV., p. 95, and *Institute of Actuaries' Text-book*, Part II.

Therefore

$$\int_0^{mh} u_x dx = h \left[u_{\frac{1}{2}h} + u_{\frac{3}{2}h} + \dots + \frac{1}{2} (u_{-\frac{1}{2}h} - u_{+\frac{1}{2}h}) \right. \\ \left. + \frac{1}{2} (u_{(m+\frac{1}{2})h} - u_{(m-\frac{1}{2})h}) \right]$$

if second differences of u_x are approximately constant.

Therefore, changing h into $2h$, we obtain

$$\int_0^{2mh} u_x dx = 2h \left[u_h + u_{3h} + \dots + \frac{1}{2} (u_{-h} - u_h) + \frac{1}{2} (u_{(2m+1)h} - u_{(2m-1)h}) \right].$$

If, then, u_x is zero for all values of x after some finite value ξ , we find that an approximate value of $\int_0^\xi u_x dx$ is

$$2h \left[u_h + u_{3h} + u_{5h} + \dots + \frac{1}{2} (u_{-h} - u_h) \right]^*.$$

9. If we write $2\phi(x) = u_x - u_{-x}$ and assume

$$\phi(x) = a_1 \frac{x}{h} + a_2 \frac{(x+h)(x-h)}{3! h^3} + \dots + a_{2r+1} \frac{(x+rh)^{(2r+1)}}{(2r+1)! h^{2r+1}}, \dots,$$

where $x^{(m)} = x(x-h)(x-2h)\dots\{x-(m-1)h\}$,

we have
$$\Delta\phi(x) = a_1 + \frac{a_2(x+h)^{(2)}}{2! h^2} + \dots,$$

$$\Delta^3\phi(x) = a_3 + \frac{a_5(x+2h)^{(2)}}{2! h^2} + \dots,$$

and in general

$$\Delta^{2r+1}\phi(x) = a_{2r+1} + \frac{a_{2r+3}\{x+(r+1)h\}^{(2)}}{2! h^2} + \dots,$$

Therefore
$$a_{2r+1} = [\Delta^{2r+1}\phi(x)]_{x=-(r+1)h}.$$

Now
$$\Delta^{2r+1}u_{-x} = u_{-[x+(2r+1)h]} - {}^{2r+1}C_1 u_{-(x+2rh)} + \dots - u_{-x}.$$

Therefore

$$[\Delta^{2r+1}u_{-x}]_{x=-(r+1)h} = u_{-rh} - {}^{2r+1}C_1 u_{-(r-1)h} \dots - u_{(r+1)h} \\ = -\Delta^{2r+1}u_{-rh}.$$

* Mr. G. F. Hardy, *Institute of Actuaries' Text-book*, Part II., Ch. XXIV.

$$\text{Therefore } a_{2r+1} = \frac{1}{2} (\Delta^{2r+1} u_{-rh} + \Delta^{2r+1} u_{-(r+1)h}).$$

Therefore

$$\begin{aligned} 2\phi(x) = & \frac{x}{h} \Delta(u_0 + u_{-h}) + \frac{(x+h)^{(3)}}{3! h^3} \Delta^3(u_{-h} + u_{-2h}) \\ & \dots + \frac{(x+rh)^{(2r+1)}}{(2r+1)! h^{2r+1}} \Delta^{2r+1}(u_{-rh} + u_{-(r+1)h}) + \dots \end{aligned}$$

If this series be added to that obtained for the value of $f(x)$, we find that

$$\begin{aligned} u_x = u_0 + \frac{x}{h} \cdot \frac{1}{2} \Delta(u_0 + u_{-h}) + \frac{x^2}{2! h^2} \Delta^2 u_{-h} \\ + \frac{x(x^2 - h^2)}{3! h^3} \frac{1}{2} \Delta^3(u_{-h} + u_{-2h}) + \dots, \end{aligned}$$

which is Stirling's interpolation formula.*

GENERALIZATION OF THE DIFFERENTIAL OPERATIVE SYMBOL WITH AN EXTENDED FORM OF BOOLE'S EQUATION

$$\mathfrak{J}(\mathfrak{J}-1)(\mathfrak{J}-2)\dots(\mathfrak{J}-n+1) = x^n \frac{d^n}{dx^n}.$$

By *Rev. F. H. Jackson, M.A.*

IT may be interesting to notice that just as the various functions $\Gamma(x)$, $F(\alpha, \beta, \gamma, x)$, $J_n(x)$, etc., correspond to more general q -form functions denoted $\Gamma_q(x)$, $\phi(\alpha, \beta, \gamma, q, x)$, $J_n(q, x)$, etc., in which numbers of the form $(q^n - 1)/(q - 1)$ replace the number n , so also the theorem relating to symbolic differential operators

$$x \frac{d}{dx} \left(x \frac{d}{dx} - 1 \right) \dots \left(x \frac{d}{dx} - n + 1 \right) = x^n \frac{d^n}{dx^n}$$

(Boole, *Phil. Trans.* 1841)

* Boole, *Calculus of Finite Differences*, Ch. III., Examples.

corresponds to a q -form differential operative theorem

$$\left(\frac{q^{x(d/dx)} - 1}{q - 1}\right) \left(\frac{q^{x(d/dx)-1} - 1}{q - 1}\right) \dots \left(\frac{q^{x(d/dx)-n+1} - 1}{q - 1}\right) = x^n \cdot \Delta^n,$$

which, in accordance with the notation $(q^a - 1)/(q - 1) = [\alpha]$ used in previous papers, may be written

$$\left[x \frac{d}{dx}\right] \left[x \frac{d}{dx} - 1\right] \dots \left[x \frac{d}{dx} - n + 1\right] = x^n \Delta^n$$

or
$$[\mathfrak{J}][\mathfrak{J} - 1] \dots [\mathfrak{J} - n + 1] = x^n \Delta^n.$$

q -Differential equations.

To show the remarkable analogy between the operative symbols $[\mathfrak{J}]$ and $\mathfrak{J}$ we note that

$$[\mathfrak{J} + \alpha][\mathfrak{J} + \beta]y - \frac{1}{x}[\mathfrak{J}][\mathfrak{J} + \gamma - 1]y = 0$$

is a q -differential equation, which among its particular integrals has

$$y = 1 + \frac{(q^\alpha - 1)(q^\beta - 1)}{(q - 1)(q^\gamma - 1)}x + \frac{(q^\alpha - 1)(q^{\alpha+1} - 1)(q^\beta - 1)(q^{\beta+1} - 1)}{(q - 1)(q^2 - 1)(q^\gamma - 1)(q^{\gamma+1} - 1)}x^2 + \dots$$

The analogy between these forms and Kummer's well-known form of the hypergeometric differential equation

$$(\mathfrak{J} + \alpha)(\mathfrak{J} + \beta)y - \frac{1}{x}\mathfrak{J}(\mathfrak{J} + \gamma - 1)y = 0$$

is obvious. It is perfectly easy to construct in this way parallel forms of other differential equations, for example

$$[\mathfrak{J} + n][\mathfrak{J} - n]y + x^2y = 0,$$

with solutions $J_n(q, x)$ $J_{-n}(q, x)$, etc., corresponding to Bessel's equation

$$(\mathfrak{J} + n)(\mathfrak{J} - n)y + x^2y = 0.$$

Equations of the above form may also be looked at as belonging to a class of functional equations.

Functional equations.

$$X_1 \phi(x) + X_2 \phi(qx) + X_3 \phi(q^2x) + \dots + X_n \phi(q^n x) = 0,$$

A general theory of the q -differential equations would really determine the solution of functional equations of the above type. For example the q -hypergeometric series is a solution of the functional equation

$$(Ax + B) \phi(q^2x) + (Cx + D) \phi(qx) + (Ex + F) \phi(x) = 0$$

for the following special values of the constants $A, B, C, \dots$

$$A = q^{a+\beta}, \quad B = -q^{\gamma-1}, \quad C = -q^a - q^\beta, \quad D = q^{\gamma-1}, \quad E = 1, \quad F = -1.$$

I do not propose in this short note to enter on any discussion of the more general classes of functional equations of the above type, but in connection with such equations we may remember a remark of Picard's, that the problems of the future will demand the solution of functional equations rather than the solution of differential equations. Perhaps the way to the solution of large classes of functional equations will be by the use of the q -form differential operative symbol, operating not only on one but on two independent variables in forms analogous to those of partial differential equations.

Generalized form of Boole's equation.

Consider an operative symbol Δ such that

$$\Delta \phi(x) = \frac{\phi(qx) - \phi(x)}{qx - x}.$$

Now, provided that $\phi(x)$ be a function possessing a derivative, we may write

$$\lim_{q=1} \Delta \phi(x) = \phi'(x).$$

Repetition of the Δ operation gives us

$$\Delta^2 \phi(x) = \frac{\phi(q^2x) - (q+1)\phi(qx) + q\phi(x)}{(q^2x - qx)(qx - x)}. \quad (1).$$

Let us now introduce an operative symbol Q , such that an operation with Q is equivalent to the substitution of qx for x in the function operated upon. Thus $Q^n \phi(x) = \phi(Q^n x)$, so that equation (1) may be written

$$\begin{aligned}\Delta^2 \phi(x) &= \frac{\{Q^2 - (q+1)Q + q\} \phi(x)}{(q^2x - qx)(qx - x)} \\ &= \frac{(Q-1)(Q-q)\phi(x)}{(q^2x - qx)(qx - x)}.\end{aligned}$$

Without complicating this paper by a mass of formal algebra we may say that it is easy to proceed to the general form

$$\Delta^n \phi(x) = \frac{\left\{ Q^n - [n]Q^{n-1} + q \frac{[n][n-1]}{[2]!} Q^{n-2} - \dots (-1)^n q^{\frac{1}{2}n(n-1)} Q^0 \right\} \phi(x)}{(q^n x - q^{n-1}x)(q^{n-1}x - q^{n-2}x) \dots (qx - x)}$$

which by the q -binomial theorem may be expressed

$$\Delta^n \phi(x) = \frac{(Q-1)(Q-q) \dots (Q - q^{n-1}) \phi(x)}{q^{\frac{1}{2}n(n-1)} (q-1)^n x^n}.$$

At this point we must investigate the nature of the symbolic operator Q . In fact we shall show that it is

$$q^{x(d/dx)}.$$

Consider

$$q^{xD} = 1 + \log_e q \left(x \frac{d}{dx} \right) + \frac{1}{2!} (\log_e q)^2 \left(x \frac{d}{dx} \right)^2 + \dots$$

Operating with this on x^m we obtain $q^m x^m$, and so with restrictions on the nature of the function we may write

$$q^{x(d/dx)} \phi(x) = \phi(qx),$$

$$(q^{xD} - 1) \phi(x) = \phi(qx) - \phi(x),$$

and a little consideration shows us that we may replace

$$q^{xD} - q \text{ by } q(q^{xD-1} - 1),$$

$$q^{xD} - q^r \text{ by } q^r(q^{xD-r} - 1),$$

so that

$$\Delta^n \phi(x) = \frac{(Q-1)(Q-q) \dots (Q - q^{n-1}) \phi(x)}{q^{\frac{1}{2}n(n-1)} (q-1)^n x^n}$$

becomes

$$\left(\frac{q^{xD}-1}{q-1}\right)\left(\frac{q^{xD-1}-1}{q-1}\right)\dots\left(\frac{q^{xD-n+1}-1}{q-1}\right)\phi(x)=x^n.\Delta^n\phi(x),$$

which conventionally has a limiting ($q=1$) form

$$xD(xD-1)\dots(xD-n+1)\phi(x)=x^nD^n\phi(x).$$

I say conventionally, because although we write

$$L_{q=1}[a]=a$$

we cannot write

$$L_{q=1}[xD] \text{ that is } L_{q=1}\frac{q^{xD}-1}{q-1}=x\frac{d}{dx},$$

except in connection with the function operated upon. Thus

$$\text{if } [xD]\phi[x]=\psi(q,x)$$

$$\text{and } L_{q=1}\psi(q,x)=\left(x\frac{d}{dx}\right)\phi(x),$$

then conventionally we may write

$$L_{q=1}[xD]=xD,$$

so that finally we have

$$[\mathfrak{J}][\mathfrak{J}-1]\dots[\mathfrak{J}-n+1]=x^n\left\{\frac{1}{x}[\mathfrak{J}]\right\}^n$$

where

$$[\mathfrak{J}]=\frac{q^{\mathfrak{J}}-1}{q-1},$$

$$\mathfrak{J}=x\frac{d}{dx}.$$

In another note I propose to discuss a q -form of Taylor's theorem in connection with this operator $[\mathfrak{J}]$.

q -FORM OF TAYLOR'S THEOREM.By *Rev. F. H. Jackson, M.A.*

It may be interesting to compare the well-known form of Taylor's theorem

$$\Phi(x) = \Phi(a) + (x-a)\Phi'(a) + (x-a)^2 \frac{\Phi''(a)}{2!} + \dots$$

with a q -form

$$\begin{aligned} \Phi(x) = & \Phi(a) + (x-a)\Delta\Phi(a) + (x-a)(x-aq)\frac{\Delta^2\Phi(a)}{[2]!} + \dots \\ & \dots + (x-a)(x-aq)\dots(x-aq^{n-1})\frac{\Delta^n\Phi(a)}{[n]!} + \dots \end{aligned}$$

in which Δ denotes the q -differential operator

$$\frac{1}{x} \left[x \frac{d}{dx} \right] = \frac{1}{x} \cdot \frac{q^{x(d/dx)} - 1}{q - 1}$$

discussed in the preceding note.

Since, in general,

$$\Delta\Phi(x) = \frac{\Phi(qx) - \Phi(x)}{qx - x},$$

we see that

$$\Delta x^n = [n] x^{n-1}.$$

Now the q analogue of the Binomial

$$(x-a)^n = x^n - nx^{n-1}a + \dots$$

is $(x-a)(x-aq)\dots(x-aq^{n-1})$

$$= x^n - [n]x^{n-1}a + q \frac{[n][n-1]}{[2]!} x^{n-2}a^2 \dots,$$

so that, by a little algebra, we obtain

$$\Delta (x-a)(x-aq)\dots(x-aq^{n-1}) = [n](x-a)\dots(x-aq^{n-2}),$$

the analogue of

$$\frac{d}{dx}(x-a)^n = n(x-a)^{n-1}.$$

If now $\Phi(x)$ be capable of expansion in the form

$$C_0 + C_1(x-a) + C_2(x-a)(x-aq) + \dots,$$

writing

$$\Phi(x) = C_0 + C_1(x-a) + \dots + C_n(x-a)\dots(x-aq^{n-1}) + \dots,$$

we have

$$\Delta \Phi(x) = C_1 + [2]C_2(x-a) + \dots + [n]C_n(x-a)\dots(x-aq^{n-2}) + \dots$$

$$\Delta^2 \Phi(x) = [2]C_2 + C_3[3][2](x-a) + \dots$$

$$\dots\dots\dots,$$

and on putting $x=a$ in this series of equations, we obtain

$$C_0 = \Phi(a),$$

$$C_1 = \Delta \Phi(a),$$

$$\dots\dots\dots,$$

$$C_n = \frac{1}{[n]!} \Delta^n \Phi(a),$$

$$\dots\dots\dots,$$

resulting in the form

$$\Phi(x) = \Phi(a) + (x-a)\Delta \Phi(a) + \dots$$

$$\dots + (x-a)(x-aq)\dots(x-aq^{n-1}) \frac{\Delta^n \Phi(a)}{[n]!} + \dots.$$

Now, for example, if $E_q(x)$ be a q -function analogous to the exponential and such that

$$\Delta . E_q(x) = E_q(x),$$

we should have

$$E_q(x) = E_q(a) \left\{ 1 + (x-a) + \frac{(x-a)(x-aq)}{[2]!} + \dots \right\},$$

which theorem is well known in a slightly different form, viz.

$$\begin{aligned} & \left(1 + \frac{x}{q-1} + \frac{x^2}{(q-1)(q^2-1)} + \dots\right) \\ & \quad \div \left(1 + \frac{a}{q-1} + \frac{a^2}{(q-1)(q^2-1)} + \dots\right) \\ & = 1 + \frac{x-a}{q-1} + \frac{(x-a)(x-aq)}{(q-1)(q^2-1)} + \dots \end{aligned}$$

in connection with the series and product which are fundamental in Jacobi's treatment of *q*-forms of Elliptic Functions. Nor is it difficult to see that as Taylor's theorem may be symbolically expressed

$$e^{xD_a}\Phi(a) = e^{aD_x}\Phi(x),$$

where D_x is the symbolic operator d/dx , so the symbolic Q form is

$$E_q(x\Delta_a)\Phi(a) = E_q(a\Delta_x)\Phi(x).$$

We note that

$$\begin{aligned} E_q(x) &= \left\{1 + \frac{x}{q}(q-1)\right\} \left\{1 + \frac{x}{q^2}(q-1)\right\} \\ &\quad \times \left\{1 + \frac{x}{q^3}(q-1)\right\} \dots ad \text{ inf.} \end{aligned}$$

$$= 1 + \frac{x}{[1]} + \frac{x^2}{[2]!} + \frac{x^3}{[3]!} + \dots,$$

$$E_{1/q}(x) = 1 + \frac{x}{[1]} + q \frac{x^2}{[2]!} + q^3 \frac{x^3}{[3]!} + \dots,$$

$$E_q(x) E_{1/q}(y) = 1 + \frac{(x+y)}{[1]} + \frac{(x+y)(x+qy)}{[2]!} + \dots,$$

$$E_q(x) E_{1/q}(-x) = 1.$$

I leave the further consideration of these forms to a paper which will deal with the solution of a general class of functional equations

$$X_1\Phi(x) + X_2\Phi(qx) + \dots + X_n\Phi(q^n x) = 0.$$

ON THE DERIVATES OF NON-DIFFERENTIABLE FUNCTIONS.

By W. H. YOUNG, Sc.D., F.R.S.

§ 1. LITTLE or nothing has been written on the theory of non-differentiable functions. Writers have contented themselves with constructing examples of such functions without enquiring whether the examples so constructed could fairly be regarded as typical or not. We may add that these examples are few in number and strike the ordinary mathematical reader as very special in character. The present note is to be regarded as a first attempt at a discussion of such functions in general. The argument, however, turns not on the functions themselves, but on their derivatives, and cannot therefore be regarded as complete. Moreover, even from the point of view adopted, it would be desirable to actually construct new examples illustrating the results here stated. These results may be thus summarised:—

If a function $f(x)$ is non-differentiable in an interval;

(1) *there is necessarily a distinction of right and left in the values of the derivatives at a set of points which is everywhere dense and is of the first category;*

(2) *the upper and lower bounds of the values of the derivatives at the points of this set are respectively $+\infty$ and $-\infty$;*

(3) *at the remaining points of the interval both the upper derivatives are $+\infty$ and both the lower derivatives are $-\infty$.*

It would therefore seem that the examples of non-differentiable functions hitherto given, if not of the most general type conceivable, possess a degree of generality considerably greater than might have been anticipated.

§ 2. LEMMA. *The derivatives of a non-differentiable function are unbounded both above and below in every interval.*

This enunciation may also be worded as follows:—*The upper and lower bounds of the derivatives of a non-differentiable function are in every interval respectively $+\infty$ and $-\infty$.*

We have to shew that if the derivatives are not unbounded both above and below the function has a differential coefficient at some point of the interval considered.

CASE 1. *Let the derivatives have both a finite upper bound and a finite lower bound in the interval considered.*

Then by a well-known theorem of Lebesgue's a differential coefficient exists at every point of a set whose content is that of the interval considered. Thus in this case the function is *not* a non-differentiable one.

CASE 2. *Let one of the bounds only be finite, say the lower bound $-L$.*

Let y be the function got by adding $(1+L)x$ to the given function $f(x)$; then the derivates of y with respect to x are got by adding $(1+L)$ to the corresponding derivates of $f(x)$, and are, therefore, always positive, being $\geq +1$. It follows that y is a monotone increasing function of x , and that therefore x is a monotone increasing function of y , say $F(y)$. The derivates of $F(y)$ at any point y' of the interval of values of y which corresponds to the interval of values of x considered, are the various limits of

$$\frac{x-x'}{y-y'}$$

as y approaches y' as limit, and therefore x approaches x' as limit. These are therefore the reciprocals of the various limits of

$$\frac{y-y'}{x-x'}$$

as x approaches x' as limit, that is to say, the various derivates of $F(y)$ are the reciprocals of derivates of y with respect to x .

Since then the derivates of y with respect to x have been shown to be all positive, being ≥ 1 , it follows that the derivates of x , *i.e.* $F(y)$ with respect to y are all positive or zero and are ≤ 1 . Hence, by Case 1, a value of y exists in the intervals of values of y considered at which a differential coefficient $F'(y)$ exists. Let this value be y' , then

$$\frac{x-x'}{y-y'}$$

has a single limit, which is either zero, or positive and ≤ 1 . In the latter case

$$\frac{y-y'}{x-x'}$$

has a single finite limit ≥ 1 ; in the former case it has no finite limit, but, since y is a monotone function of x , it has always the same sign where it is not zero, so that it has a definite infinite limit, which in the case before us is $+\infty$,

y being a monotone increasing function of x . Thus in either case

$$\frac{y - y'}{x - x'}$$

has a single limit, finite or infinite; that is to say y has a differential coefficient with respect to x . Thus in this case y , and therefore also $f(x)$, is *not* a non-differentiable function of x .

When it is the upper bound that is finite, the above reasoning requires only obvious verbal changes, so that when only one of the bounds is finite the function is *not* a non-differentiable one.

The only case remaining being that in which the upper bound is $+\infty$ and the lower bound is $-\infty$, it follows that the Lemma is proved.

COR. Hence with the notation of my paper* "*On the distinction of right and left at points of discontinuity*," $\phi_R, \phi_L, \psi_R, \psi_L$ being the associated functions of any one of the derivates of a non-differentiable function $f(x)$, we have at every point

$$\phi_R = \phi_L = +\infty,$$

$$\psi_R = \psi_L = -\infty,$$

§ 3. THEOREM 1. If $f(x)$ is a non-differentiable function, at every point of the interval considered which does not belong to a certain set of the first category both the upper derivates are $+\infty$ and both the lower derivates are $-\infty$.

For† except at a certain set of the first category the upper derivates are both upper semi-continuous and the lower derivates are both lower semi-continuous, and therefore, by the preceding Corollary, both upper derivates are $+\infty$ and both lower derivates are $-\infty$. Q.E.D.

It should be noticed that this result is in accordance with another result of the paper just quoted, viz. there is no distinction of right and left with regard to derivates except possibly at a set of the first category.‡

§ 4. THEOREM 2. The exceptional set of the first category at which there is a distinction of right and left exists and is dense everywhere. To prove this we require the First Theorem

* *Quart. Jour. of Math.*, vol. XXXIX., pp. 67-83.

† W. H. Young, "Oscillating Successions of Continuous Functions," *Proc. L.M.S.*, Series 2, vol. VI., pp. 303-304.

‡ *Loc. cit.*, p. 305.

of the Mean in an extended form: * *If there is no distinction of right and left with regard to the derivates of $f(x)$, then there is a point in the completely open interval (a, b) at which $f(x)$ has a differential coefficient, and the value of that differential coefficient is precisely*

$$m(a, b) = \frac{f(b) - f(a)}{b - a},$$

that is, for some value of θ ,

$$m(a, b) = f' \{a + \theta(b - a)\}, \quad (0 < \theta < 1).$$

Now if the exceptional set were not dense everywhere there would, in some interval, be no distinction of right and left, and therefore, by the First Theorem of the Mean in the extended form, there would be a point at which there was a differential coefficient, so that $f(x)$ would *not* be a non-differentiable function. Thus the exceptional set certainly exists and is dense everywhere.

§ 5. THEOREM 3. *The derivates of a non-differentiable function at the points of the exceptional set at which there is a distinction of right and left have no finite upper or lower bound.*

For suppose, if possible, that the derivates at the points of the exceptional set have a finite upper bound B , and let C be greater than B . Then, by the Lemma of § 2, the upper bound of the derivates in the whole interval considered is $+\infty$, and therefore by a well-known theorem this is the same as the upper bound of the incrementary ratio

$$m(x, x') = \frac{f(x) - f(x')}{x - x'}$$

when x and x' are restricted to lie in the same interval. But the incrementary ratio assumes every value between its upper and lower bounds,† so that we must be able to find a pair of points (x, x') , such that

$$m(x, x') = C.$$

We now require the following theorem, which has been proved in the paper already quoted in § 4:—*There is in the*

* See a forthcoming paper by W. H. and G. Clisholm Young in the *Quarterly Journal of Mathematics*.

† A formal proof of this elementary fact is given in the forthcoming paper previously referred to.

completely open interval (x, x') a point at which one of the lower derivates is $\geq m(x, x')$ while the other upper derivate is $\leq m(x, x')$. Since $m(x, x')$ is here $> B$, the point referred to must be one at which there is no distinction of right and left, and therefore both lower derivates must be $\geq C$ and both upper derivates $\leq C$, which is only possible if all the derivates are $= C$. But this is not possible, since there is no point at which all the derivates are equal, $f(x)$ being a non-differentiable function. Thus the original supposition is untenable, which proves that the upper bound of the values of the derivates at points of the exceptional set is $+\infty$. Similarly the lower bound is $-\infty$, which proves the theorem.

§ 6. The classical example of a non-differentiable function is Weierstrass's function

$$f(x) = \sum_{m=1}^{\infty} y_m,$$

where

$$y_m = b^m \cos a^m \pi x,$$

and

$$ab > \frac{3}{2}\pi + 1.$$

Weierstrass proved that at every point one of the upper derivates was $+\infty$ while the lower derivate on the other side was $-\infty$, and hence deduced that the function was non-differentiable. Though Weierstrass does not prove more than this, it must, of course, be possible to prove directly that by the above general theory both the upper derivates are $+\infty$ and both the lower derivates are $-\infty$ except at the points of a set of the first category. The special characteristic of Weierstrass's example would appear to consist in the fact that at all the points of the exceptional set two of the four derivates are actually $+\infty$ and $-\infty$ respectively, whereas the general theory only shows that all four derivates are unbounded. It would be of interest to enquire whether in the Weierstrass example the exceptional set is the most general set of the first category or whether it is only countable. I may perhaps have an opportunity of returning to this matter on a subsequent occasion.

NOTES ON INTEGRAL EQUATIONS.

II.

By *H. Bateman.**The method of least squares.*

§ 1. AN integral equation of the first kind

$$f(s) = \int_a^b \kappa(s, t) \phi(t) dt \dots\dots\dots (1),$$

in which $f(s)$ and $\kappa(s, t)$ are supposed to be known for a certain set of values of s and $\phi(t)$ is to be calculated, can be likened to a system of m linear equations*

$$\left. \begin{aligned} n_1 &= a_1 x + b_1 y + c_1 z + \dots k_1 t \\ n_2 &= a_2 x + b_2 y + c_2 z + \dots k_2 t \\ \dots\dots\dots \\ n_m &= a_m x + b_m y + c_m z + \dots k_m t \end{aligned} \right\} \dots\dots\dots (2),$$

between the μ unknown quantities $x, y, z, \dots, t$.

The most interesting case occurs when the homogeneous equation

$$0 = \int_a^b \kappa(s, t) \chi(t) dt \dots\dots\dots (3)$$

cannot be satisfied by a continuous function $\chi(t)$ which is not identically zero. The function $\kappa(s, t)$ is then said to be *perfect* for the given set of values of s and the range of integration (a, b) .

If the function $\phi(t)$ occurring in equation (1) is restricted to be continuous the function $f(s)$ must possess certain definite properties, usually there will be a number of linear conditions to be satisfied, which will be, in general, of the type

$$\int_c^d \omega(s) f(s) ds = 0, \text{ where } \int_c^d \omega(s) \kappa(s, t) ds \equiv 0 \dots (4).$$

We may regard this as corresponding with the case in which the number of linear equations (2) is greater than the number of unknown quantities. For we know that under these circumstances the exact solution of equations (2) is impossible unless they are not all independent. This exceptional case occurs when we have a number of relations of the type

$$\sum_1^m \omega_r n_r = 0, \text{ where } \sum_1^m a_r \omega_r = \sum_1^m b_r \omega_r = \dots \sum_1^m k_r \omega_r = 0 \dots (5).$$

* This analogy is followed up in a paper by the author 'On the inversion of a definite integral,' *Proceedings of the London Mathematical Society*, Ser. 2, vol. 1v, Parts 6 and 7, p. 461.

Now the method of least squares as applied to the system of linear equations (2) deals with the case in which the equations of condition just written down are not satisfied. The number of equations is supposed to be far greater than the number of unknown quantities and the values of the quantities $n_1 \dots n_m$ being obtained by means of observations are liable to small errors. The exact solution of equations (2) being then out of the question we are reduced to finding the most probable values of the quantities $x, y, \dots t$ and we may regard these as the ones which will give the best approximations to the quantities $n_1 \dots n_m$, when substituted in the system of equations (2).

The analogous case in the theory of integral equations which must be treated in the same way occurs when it is not possible to obtain an *exact* representation of a given function $f(s)$ by a definite integral of the type (1). This may be due to the fact that $f(s)$ does not satisfy the linear conditions (4) which are characteristic of all functions represented by the formula (1). The method we are about to explain, however, does not presuppose a knowledge of these conditions. It can be applied at once to the case in which there is some uncertainty as to whether $f(s)$ does not satisfy all the conditions that are required. If the conditions are satisfied the method leads to the exact representation of $f(s)$ in the given form. If the conditions are not satisfied it leads to a value of $\phi(t)$ which gives the *best possible approximation* to $f(s)$ by means of a definite integral of the form (1).*

In the method of least squares† the most probable values of the quantities $x, y, \dots, t$ are obtained by solving the system of *normal equations*

$$\left. \begin{aligned} [aa]x + [ab]y + \dots [ak]t &= [an] \\ [ba]x + [bb]y + \dots [bk]t &= [bn] \\ [ka]x + [kb]y + \dots [kk]t &= [kn] \end{aligned} \right\} \dots\dots\dots(6),$$

where $[ab]$ denotes $a_1b_1 + a_2b_2 + \dots a_mb_m$.

If $v_r = a_r x + b_r y + \dots k_r t - n_r$ these are the values of $x, y, \dots, t$ which make the sum of the squares of the residuals v_r , viz.

$$[vv] = v_1^2 + v_2^2 + \dots v_m^2$$

* A method of obtaining an approximation to a given function by means of a definite integral of a specified type is discussed in two papers by the author, *Proceedings of the London Mathematical Society*, Ser. 2, vol. iv., p. 103. *Mathematische Annalen*, Band. 63, Heft. 4, p. 528.

† Cf. Gauss, *Werke*, vol. vii., pp. 237–241; Chauvenet's *Astronomy*, vol. ii., pp. 469–558, or Glaisher's *Solutions of the Cambridge Problems and Riders for* 1878, p. 162.

a minimum. This variation problem can be supposed to give the most probable values of the quantities $x, \dots, t$ provided the observed values of the quantities $v_1, v_2, \dots, v_m$ are equally likely to be accurate.

If more importance is to be attached to some of the equations (1) than to the others, as for instance when the quantities n_r are more difficult to determine accurately for some values of r than for others, the most probable values of the quantities $x, y, \dots, t$ are obtained by making the expression

$$\sum_{r=1}^m p_r v_r^2$$

a minimum, where p_r is a positive quantity which is taken as a measure of the importance of the r^{th} equation.* The normal equations are of the same as before except that now

$$[ab] \equiv p_1 a_1 b_1 + p_2 a_2 b_2 + \dots + p_m a_m b_m.$$

Adopting an analogous method in the case of the integral equation (1) we multiply by $p(s) \kappa(s, x)$ and integrate over the set of values of s for which $f(s)$ is given. This gives the normal equation†

$$u(x) = \int_a^b g(x, t) \phi(t) dt \dots \dots \dots (7),$$

$$\text{where} \quad u(x) = \int f(s) p(s) \kappa(s, x) ds \dots \dots \dots (8)$$

$$\text{and} \quad g(x, t) = \int p(s) \kappa(s, x) \kappa(s, t) ds \dots \dots \dots (9).$$

The function $p(s)$ is supposed to be positive for all values of s belonging to the given set. Its choice depends upon the exact nature of the agreement which is desired between the definite integral (1) and the given values of $f(s)$. If it is important that this agreement should become closer as s increases, then $p(s)$ should increase with s according to the law ascribed. If the same degree of approximation is required for all values of s the function $p(s)$ may be taken to be unity. If special stress is laid upon the closeness of the agreement for a particular value of s it may be necessary to allow $p(s)$ to become infinite for this value.

It can easily be seen that the function $g(x, t)$ given by formula (9) is, in general, a definite function.‡ It is evidently

* It is called the weight of the r^{th} observation.

† Cf. *British Association Reports* (1907), Section A, p. 447.

‡ See a paper by the author, *Messenger of Mathematics*, vol. XXXVII, pp. 91-95.

a symmetric function of x and t , and, if $\alpha(x)$ is any continuous function,

$$\int_a^b \int_a^b g(x, t) \alpha(x) \alpha(t) dx dt = \int [A(s)]^2 ds,$$

where $A(s) = \sqrt{\{p(s)\}} \cdot \int_a^b \kappa(s, x) \alpha(x) dx$.

Now $A(s)$ is not identically zero, since $\kappa(s, x)$ is *perfect*, it is real since $p(s)$ is positive, consequently the double integral is positive, and so $g(x, t)$ is a definite function.

Since $g(x, t)$ is a definite function the solution of the integral equation (7) is unique. Of course we are not sure that the equation does possess a continuous solution $\phi(t)$. The question depends upon whether or not we can find a continuous function $\phi(t)$ which will make the integral

$$V = \int p(s) \left[f(s) - \int_a^b \kappa(s, t) \phi(t) dt \right]^2 ds \dots\dots (10)$$

a minimum, the integration being taken over the given set of values of s .

To justify this statement let us suppose that a minimum exists and that $\phi(t)$ is the corresponding function, then corresponding to a small variation $\delta\phi(t)$, we have

$$\begin{aligned} \delta V &= 2 \int p(s) \left[\int_a^b \int_a^b \kappa(s, x) \kappa(s, t) \phi(t) \delta\phi(x) dx dt \right. \\ &\quad \left. - \int_a^b f(s) \kappa(s, t) \delta\phi(x) dx \right] ds \\ &= 2 \int_a^b \left[\int_a^b g(x, t) \phi(t) dt - u(x) \right] \delta\phi(x) dx. \end{aligned}$$

In order that δV may be zero for all variations $\delta\phi$ we must have

$$u(x) = \int_a^b g(x, t) \phi(t) dt,$$

the normal equation under consideration.

The general theory of the integral equation (1) must eventually depend upon the problem in the calculus of variations of finding a function $\phi(t)$ which will make the expression (10) a minimum. If this minimum value is zero the function obtained is a solution of the integral equation.

When $\kappa(s, t)$ is an "allgemeiner Kern" (Hilbert, *Göttinger Nachrichten* (1904), Heft. I.) a minimum apparently does not always exist, for according to the definition of an allgemeiner Kern, to every small positive quantity ϵ and

every continuous function $f(s)$ there corresponds a function $\phi(t)$ for which

$$\int_a \left[f(s) - \int_a^b \kappa(s, t) \phi(t) dt \right]^2 ds < \epsilon.$$

A true minimum value for the integral in this case only exists if the integral equation (1) can be actually solved and then the minimum value is zero.

§2. If the function $f(s)$ satisfies all the conditions of the type (4) where $\omega(s)$ is a continuous function we can show that a continuous function $\phi(t)$ which satisfies (7) is a solution of the integral equation (1). For, if we put

$$h(s) = f(s) - \int_a^b \kappa(s, t) \phi(t) dt \dots \dots \dots (11)$$

we have from (7), (8), (9)

$$\int h(s) p(s) \kappa(s, x) ds = 0 \dots \dots \dots (12).$$

This is a linear condition of type (4) with $\omega(s) = h(s) p(s)$, hence $f(s)$ must satisfy

$$\int h(s) p(s) f(s) ds = 0.$$

Substituting the values of $f(s)$ from (11) and making use of (12) we obtain

$$\int [h(s)]^2 p(s) ds = 0.$$

Now $p(s)$ is positive, therefore $h(s)$ must be identically zero, and equation (11) indicates that $\phi(t)$ is a solution of (1).

§3. A second minimum problem is suggested by Gauss' analysis (Cf. Bertrand, *Calcul des Probabilités*, Ch. XI, p. 290).

Let τ be an assigned value of the variable t . We propose to determine, if possible, an integrable function $\alpha(s, \tau)$ so that, with the notation of equation (11),

$$\int h(s) \alpha(s, \tau) ds = \beta(\tau) - \phi(\tau) \dots \dots \dots (13),$$

and

$$\int [\alpha(s, \tau)]^2 ds$$

is a minimum. The function $\beta(\tau)$ is supposed to be entirely independent of the form of $\phi(t)$.

$$\begin{aligned}\text{Put} \quad \xi(x) &= \int h(s) \kappa(s, x) ds \\ &= -\int_a^b g(x, t) \phi(t) dt + u(x),\end{aligned}$$

where now $p(s) \equiv 1$, and let us suppose that this equation can be solved for $\phi(t)$, so that

$$\phi(t) = -A.\xi(t) + A.u(t),$$

where A is a distributive operator.

$$\text{Now let} \quad \beta(s, t) = A.\kappa(s, t) \dots \dots \dots (14),$$

$$\text{then} \quad A.\xi(t) = \int h(s) \beta(s, t) ds,$$

$$\begin{aligned}\text{and so} \quad \int h(s) \beta(s, t) ds &= A.u(t) - \phi(t) \\ &= v(t) - \phi(t),\end{aligned}$$

where $v(t)$ is independent of the form of $\phi(t)$. Hence $\beta(s, \tau)$ is a particular function of the type $\alpha(s, \tau)$. Let $\alpha(s, \tau)$ be any integrable function which satisfies (13), then

$$\int h(s) [\alpha(s, \tau) - \beta(s, \tau)] ds = \beta(\tau) - v(\tau).$$

This being true for all forms of $\phi(t)$, we must have

$$\int \kappa(s, t) [\alpha(s, t) - \beta(s, \tau)] ds = 0 \dots \dots \dots (15),$$

$$\int f(s) [\alpha(s, \tau) - \beta(s, \tau)] ds = \beta(\tau) - v(\tau).$$

Operating on equation (15) with the operator A , we obtain

$$\int \beta(s, t) [\alpha(s, \tau) - \beta(s, \tau)] ds = 0.$$

Putting $t = \tau$, this gives

$$\int \beta(s, \tau) [\alpha(s, \tau) - \beta(s, \tau)] ds = 0$$

$$\text{or} \quad \int [\alpha(s, \tau)]^2 ds = \int [\beta(s, \tau)]^2 ds + \int [\alpha(s, \tau) - \beta(s, \tau)]^2 ds.$$

Since the second term on the right-hand side is always positive, the integral

$$\int [\alpha(s, \tau)]^2 ds$$

will have its minimum value when $\alpha(s, \tau) = \beta(s, \tau)$.

The function which gives the minimum value is thus obtained by solving the equation

$$\kappa(s, t) = \int_a^b \beta(s, x) u(x, t) dx,$$

and taking the value of $\beta(s, x)$ for $x = \tau$.

It is to be remarked that $\beta(s, x)$ is not, in general, a *continuous* function, if it were, we could put

$$f(s, \sigma) = \int_a^b \beta(s, x) \kappa(\sigma, x) dx,$$

$$\kappa(s, t) = \int f(s, \sigma) \kappa(\sigma, t) d\sigma.$$

The last equation is a homogeneous integral equation of the second kind for $\kappa(s, t)$, and since $f(s, \sigma)$ is continuous there are only a *finite* number of linearly independent functions $\psi(s)$ for which the equation

$$\psi(s) = \int f(s, \sigma) \psi(\sigma) d\sigma$$

can be satisfied. Hence, unless $\kappa(s, t)$ can be expressed as a finite sum of the form

$$\kappa(s, t) = \sum_{n=1}^m \psi_n(s) \chi_n(t),$$

the above equation is impossible.

THE INTEGRAL OF THE PRODUCT OF THREE ZONAL HARMONICS.

By D. Edwardes, B.A.

THE value of this integral was found originally by Adams and Ferrers, and to obtain it by a deductive method was the main object in the analytical part of W. D. Niven's memoir *On certain definite integrals occurring in Spherical Harmonic Analysis, &c.* (*Phil. Trans.*, Part I., 1879). The author obtains a development of $\iint V dS$, taken over the surface of a sphere in a series of powers of Laplace's operator. This operator is transformed, symbols such as d/dx being first broken up into $d/dx_1, d/dx_2, d/dx_3$, operating upon separate factors in the integrand. The result is obtained at first in the form of a series. The whole difficulty, however, consists in expressing the product $P_m P_n$ as a sum of harmonics. When this has been effected, the required result follows at once from the known integrals $\int P_r P_s dx$ and $\int P_r^2 dx$. If in the product

$P_m P_n$, m, n are either both even or both odd, only harmonics of even order can occur, and only harmonics of odd order in the remaining case. The harmonic of highest order is clearly P_{m+n} , and it will be seen in the course of the following work that the one of lowest order is P_{m-n} . The following method is based upon the differential equation for the product $P_m P_n$.

If $y = P_m(x) P_n(x) \dots \dots \dots (1)$,
then, using the equation

$$\frac{d}{dx} (1-x^2) \frac{dP_i}{dx} + i(i+1) P_i = 0 \dots \dots \dots (2),$$

we readily find

$$\begin{aligned} \frac{d^2}{dx^2} (1-x^2) \frac{d}{dx} (1-x^2) \frac{dy}{dx} + 2\lambda \frac{d}{dx} (1-x^2) \frac{dy}{dx} \\ - 2\lambda x \frac{dy}{dx} + \{(n.n+1 - m.m+1)^2 - 2\lambda\} y = 0 \dots (3), \end{aligned}$$

where $\lambda = n(n+1) + m(m+1) \dots \dots \dots (4)$.

We cannot assume $y = \Sigma A_r P_r$ in this equation, for the reason that dP_r/dx , and therefore also xdP_r/dx , cannot be expressed otherwise than by a long train of harmonics, and this would not give a relation between two consecutive A 's. We assume

$$y = \Sigma A_r \{\alpha_{r+2} P_{r+2} - \alpha_r P_{r+1}\} \dots \dots \dots (5),$$

where $\alpha_r = r(r+1) - \lambda$
 $= r(r+1) - n(n+1) - m(m+1) \dots \dots (6)$.

Let $f(r) = r^2(r+1)^2 + 2(1-\lambda)r(r+1)$
 $+ (n.n+1 - m.m+1)^2 - 2\lambda \dots (7)$.

Substituting the general term in (5), using (2) and omitting for the moment the coefficients A_r , the left-hand member of (3) becomes

$$\alpha_{r+2} f(r) P_r - \alpha_r f(r+2) P_{r+2} + 2\alpha_r \alpha_{r+2} \left\{ x \frac{dP_r}{dx} - x \frac{dP_{r+2}}{dx} \right\} \dots (8).$$

By the known formulæ

$$\frac{dP_{r+2}}{dx} - \frac{dP_r}{dx} = (2r+3) P_{r+1} \dots \dots \dots (C),$$

and $(2r+3)xP_{r+1} = (r+2)P_{r+2} + (r+1)P_r$

we have $x \left(\frac{dP_{r+2}}{dx} - \frac{dP_r}{dx} \right) = (r+2)P_{r+2} + (r+1)P_r$.

Hence (8) becomes

$$\alpha_{r+2} P_r \{f(r) - 2(r+1)\alpha_r\} - \alpha_r P_{r+2} \{f(r+2) + 2(r+2)\alpha_{r+2}\} \dots \dots \dots (9).$$

We have

$$\begin{aligned} & f(r) - 2(r+1)\alpha_r \\ &= r^2(r+1)^2 + 2(1-\lambda)r(r+1) + (n.n+1-m.m+1)^2 - 2r(r+1)^2 \\ &= r^4 - (2\lambda+1)r^2 + (n.n+1-m.m+1)^2 \\ &= r^4 - \{2n(n+1) + 2m(m+1) + 1\}r^2 + (n.n+1-m.m+1)^2 \\ &= \{r^2 - (m+n+1)^2\} \{r^2 - (n-m)^2\} \dots \dots \dots (10). \end{aligned}$$

Also

$$\begin{aligned} & f(r+2) + 2(r+2)\alpha_{r+2} \\ &= (r+2)^2(r+3)^2 + 2(r+2)(r+3) + 2(r+2)^2(r+3) \\ &= 2\lambda\{(r+2)(r+3) + r+3\} + (n.n+1-m.m+1)^2 \\ &= (r+2)(r+4)(r+3)^2 - 2\lambda(r+3)^2 + (n.n+1-m.m+1)^2 \\ &= (r+3)^4 - (2\lambda+1)(r+3)^2 + (n.n+1-m.m+1)^2 \\ &= \{(r+3)^2 - (m+n+1)^2\} \{(r+3)^2 - (n-m)^2\} \dots \dots \dots (11). \end{aligned}$$

Putting these values in (9), restoring the A 's, and equating to zero the whole coefficient of P_r , we get

$$\begin{aligned} & A_r \alpha_{r+2} \{r^2 - (m+n+1)^2\} \{r^2 - (n-m)^2\} \\ &= A_{r-2} \alpha_{r-2} \{(r+1)^2 - (m+n+1)^2\} \{(r+1)^2 - (n-m)^2\} \dots (12). \end{aligned}$$

Now if m and n are either both even or both odd, r must be even, and must be odd in the remaining case. Hence no zero factor can be of the form

$$r^2 - (m+n+1)^2 \text{ or } (r+1)^2 - (n-m)^2.$$

It follows that all the A 's with suffixes lower than $n-m$ are zero and that the series terminates when $r = m+n-2$.

$$\text{Hence } y = \sum_{r=m-n}^{r=m+n-2} A_r \{\alpha_{r+2} P_r - \alpha_r P_{r+2}\} \dots \dots \dots (13).$$

Equating coefficients of x^{m+n} in this, we get

$$-\alpha_{m+n-2} A_{m+n-2} = \frac{2m! 2n! (m+n)! (m+n)!}{(2m+2n)! m! m! n! n!} \dots \dots (14),$$

and writing $m+n-2s$ for r in (12),

$$\begin{aligned} & \alpha_{m+n-2s-2} A_{m+n-2s-2} \\ &= A_{m+n-2} \frac{\alpha_{m+n} \alpha_{m+n-2}}{\alpha_{m+n-2s}} \cdot \frac{\phi(m+n-2) \dots \phi(m+n-2s)}{\phi(m+n-1) \dots \phi(m+n-2s+1)}, \end{aligned}$$

where $\phi(r) = \{r^2 - (m+n+1)^2\} \{r^2 - (n-m)^2\}$.

We have $\phi(m+n-2s+1) - \phi(m+n-2s)$
 $= 4(2m+2n-4s+1)\{2s^2-s(2m+2n+1)+mn\}$
 $= 2(2m+2n-4s+1)\alpha_{m+n-2s}$, by (6), ..., (15).

Hence $A_{m+n-2s}\alpha_{m+n-2s+2} - A_{m+n-2s-2}\alpha_{m+n-2s-2}$
 $=$ the whole coefficient of P_{m+n-2s} in (5)
 $= A_{m+n-2}\alpha_{m+n-2}\alpha_{m+n}$
 $\times \frac{2(2m+2n-4s+1)\phi(m+n-2)\dots\phi(m+n-2s+2)}{\phi(m+n-1)\dots\phi(m+n-2s+1)},$
 where $\frac{\phi(m+n-2)\phi(m+n-4)\dots\phi(m+n-2s+2)}{\phi(m+n-1)\phi(m+n-3)\dots\phi(m+n-2s+1)}$
 $= - \frac{\{(2m+2n-1)\dots(2m+2n-2s+3).3.5\dots(2s-1)\}}{\{(2m+2n)\dots(2m+2n-2s+2).2.4\dots2s\}} \times \frac{(2n-2)\dots(2n-2s+2)}{\{(2m-1)\dots(2m-2s+1).(2n-1)\dots(2n-2s+1)\}}$
 $= - \frac{(2m+2n)!(m+n-s)!(m+n-s)!}{(2m+2n-2s+1)!2^{2s}(m+n)!(m+n)!} \times \frac{2s!}{2^{2s}s!s!}$
 $\times \frac{2^{4s-4}(m-1)!(m-1)!(n-1)!(n-1)!(2m-2s)!(2n-2s)!}{(2m-1)!(2n-1)!(m-s)!(m-s)!(n-s)!(n-s)!}.$

Also, by (6), $\alpha_{m+n} = 2mn$, so that

$$\alpha_{m+n} \frac{2m!2n!(m-1)!(m-1)!(n-1)!(n-1)!}{2^4.(2m-1)!(2n-1)!m!m!n!n!} = \frac{1}{2}.$$

Hence, and by (14), (15), we get

$$A_{m+n-2s}\alpha_{m+n-2s+2} - A_{m+n-2s-2}\alpha_{m+n-2s-2}$$

$$= \frac{2m+2n-4s+1}{2m+2n-2s+1} \cdot \frac{(m+n-s)!(m+n-s)!2s!(2m-2s)!(2n-2s)!}{(2m+2n-2s)!s!s!(m-s)!(m-s)!(n-s)!(n-s)!}.$$

Thus (13) can be written

$$P_m P_n = \sum_{s=0}^{s=m} B_s P_{m-n-2s} \dots \dots \dots (16),$$

where

$$B_s = \frac{2m+2n-4s+1}{2m+2n-2s+1} \cdot \frac{(m+n-s)!(m+n-s)!2s!(2m-2s)!(2n-2s)!}{(2m+2n-2s)!s!s!(m-s)!(m-s)!(n-s)!(n-s)!}$$

.....(17).

To compare with Niven's result, § 14, p. 401, which is

$$P_r P_s = A_0 P_{r-s} + A_1 P_{r-s+2} + \dots + A_m P_{r-s+2m} + \dots + A_s P_{r+s},$$

where

$$A_m = \frac{2r-2s+4m+1}{2r+2m+1} \times \frac{(r+m)!(r+m)!2m!(2r-2s+2m)!(2s-2m)!}{(2r+2m)!m!m!(r-s+n)!(r-s+m)!(s-m)!(s-m)!},$$

we must put $m = s - \sigma$, so that $r - s + 2m = r + s - 2\sigma$. If we then write n, m, s for r, s, σ respectively, the result is identical with (17). Niven remarks (p. 398) that his result agrees with that obtained by Adams.

Multiplying both sides of (16) by P_{m+n-2s} and integrating between limits -1 and 1 ,

$$\int_{-1}^1 P_m(x) P_n(x) P_{m+n-2s}(x) dx = \frac{2}{2m+2n-4s+1} B_s.$$

$$\text{or} \quad \int_{-1}^1 P_m(x) P_n(x) P_p(x) dx = \frac{2}{m+n+p+1}$$

$$\times \frac{\{\frac{1}{2}(m+n+p)\}! \{\frac{1}{2}(m+n+p)\}! (m+n-p)! (m-n+p)! (n-m+p)!}{\left\{ \begin{array}{l} (m+n+p)! \{\frac{1}{2}(m+n-p)\}! \{\frac{1}{2}(m+n-p)\}! \{\frac{1}{2}(m-n+p)\}! \\ \times \{\frac{1}{2}(m-n+p)\}! \{\frac{1}{2}(n-m+p)\}! \{\frac{1}{2}(n-m+p)\}! \end{array} \right\}} \dots\dots(18).$$

If $m+n+p$ is odd, the integral vanishes, since

$$P_i(-x) = (-1)^i P_i(x).$$

Since Legendre's coefficients of the second kind satisfy the same relations (C) above, the work would be the same as far as (13).

The function $P_m P_m$ is a particular solution of the equation (3). The other particular solutions are $Q_m Q_n$, $P_m Q_n$, $P_n Q_m$, except when $n = m + 1$, in which case $P_m Q_{m+1}$ and $P_{m+1} Q_m$ differ only by a constant.* When $n = m + 2$ these quantities differ by a multiple of x † and Cx should be a solution, which is easily verified.

* Forsyth's *Treatise on Differential Equations*, 2nd Ed., p. 180.

† Mathematical Tripos, Part II., 1894, and in Whittaker's *Course of Modern Analysis*.

DIOPHANTINE, &c., FACTORISATION OF QUARTANS.

By Lt.-Col. Allan Cunningham, R.E., Fellow of King's College, London.

[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for help in reading the Proof-sheets, and for useful suggestions].

[This Paper is a sequel to the author's recent Paper* on *High Quartan Factorisations and Primes*: The Articles and Results herein are numbered in continuation of those in that Paper, and the numbered references herein refer to that Paper].

24. Introduction. This Paper contains (1) an account (Art. 27–37a) of several special forms of factorisable *Quartan* numbers (N and $\frac{1}{2}N$), *i.e.* of

$$N = x^4 + y^4 \text{ [} x, y \text{ not both even]}, \quad \frac{1}{2}N = \frac{1}{2}(x^4 + y^4), \text{ [} x, y \text{ both odd]... (44),}$$

and (2) the solution (Art. 38–39h) of a number of 4-tic problems, and (3) the development (Art. 40 *et seq.*) of a general Method† of factorisation of such *Quartans* depending on the reduction of N or $\frac{1}{2}N$ to one of the 2^{ic} forms

$$N \text{ or } \frac{1}{2}N = P^2 - Q^2 = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \dots\dots\dots (45),$$

—where the $(P, Q), (a, b), (c, d), (e, f), (f', e')$ are *arithmetic* partitions, different from the similar known algebraic partitions of N or $\frac{1}{2}N$ —by the solution of certain 2^{ic} Diophantine Equations of type

$$Rx^2 + Sy^2 = z^2 \dots\dots\dots (46).$$

Each such Diophantine—if soluble at all—will yield an infinity of solutions (x, y, z) , and each such solution yields a factorisable *Quartan*. This evidently opens up a great field of factorisation of large *Quartans*.

24a. Dimorph 2^{ic} Partitions. The algebraic 2^{ic} partitions of N or $\frac{1}{2}N$ will be written as before (Art. 4)—in Roman letters, and the arithmetic partitions isomorph therewith will be written with the similar italic letters, thus

$$N \text{ or } \frac{1}{2}N = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \text{ [algebraical]... (47a),}$$

$$= a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \text{ [arithmetical]... (47b).}$$

* See *Messenger of Mathematics*, Vol. XXXVI., 1907; pp. 145–174.

† The Method itself is really one of great generality, applicable to many kinds of 4^{ic} functions, *e.g.* to $(x^4 + y^4), (x^5 \mp y^5) \div (x \mp y), (x^6 + y^6) \div (x^2 + y^2)$, &c. It is hoped to give several of these applications in full detail in later Papers (now nearly ready). The Method was described by the author at the London Math. Society Meeting of 11 January, 1900; but has not as yet been published *in extenso*.

The second set can of course only exist when N or $\frac{1}{2}N$ is composite (see Art. 4); and any one of the dimorph pairs will suffice for factorisation of N or $\frac{1}{2}N$ by known Rules (Art. 26).

25. Semialgebraic Factors. The co-factors of N or $\frac{1}{2}N$ resulting from this process will be styled *Semialgebraic Factors*, and will be denoted by L, M , so that

$$N \text{ or } \frac{1}{2}N = P^2 - Q^2 = L.M; \quad L = P - Q, \quad M = P + Q; \quad L \& M = \varepsilon\varpi + 1 \dots (48),$$

They are of course expressible—being factors of N or $\frac{1}{2}N$ —in the $(a, b), (c, d), (e, f), (e', f')$ 2^{ic} forms, but (as a rule) only arithmetically (not algebraically).

26. Factorisation Rules. When any of the 2^{ic} partitions (47*b*) have been formed, then N or $\frac{1}{2}N$ can be written—[by aid of (47*a*)—in one of the forms

$$N \text{ or } \frac{1}{2}N = P^2 - Q^2 \\ = \frac{(ab)^2 - (cb)^2}{b^2 - b^2} = \frac{(cd)^2 - (cd)^2}{d^2 - d^2} = \frac{(ef)^2 - (ef)^2}{f^2 - f^2} = \frac{(f'e')^2 - (f'e')^2}{f'^2 - f'^2} \dots (49a),$$

wherein the denominators can be changed respectively into

$$(a^2 - a^2), \quad \frac{1}{2}(c^2 - c^2), \quad \frac{1}{2}(e^2 - e^2), \quad \frac{1}{2}(e'^2 - e'^2) \dots (49b),$$

and the factorisation of N or $\frac{1}{2}N$ into two co-factors is here obvious (for each numerator, as well as each denominator, is a difference of squares).

The formulæ (49*a*), though simple and direct, have the practical inconvenience of involving the *products* of $a, b, \&c.$, and thus require the computation of numbers $> N$ or $\frac{1}{2}N$ itself. The following formulæ are free from this practical defect. Let

$$L = a_1^2 + \beta_1^2 = \gamma_1^2 + 2\delta_1^2 = \eta_1^2 - 2\phi_1^2 = 2\phi_1'^2 - \eta_1'^2 \dots (50a),$$

$$M = a_2^2 + \beta_2^2 = \gamma_2^2 + 2\delta_2^2 = \eta_2^2 - 2\phi_2^2 = 2\phi_2'^2 - \eta_2'^2 \dots (50b),$$

Then the *ratios* of the two " 2^{ic} parts" of each partition are given* by either of the values below

$$\frac{a_1}{\beta_1} = \frac{a + a}{b - b} = \frac{b + b}{a - a}, \quad \frac{a_2}{\beta_2} = \frac{a + a}{b + b} = \frac{b - b}{a - a} \dots (51a),$$

$$\frac{\gamma_1}{\delta_1} = \frac{c + c}{d - d} = 2\frac{d + d}{c - c}, \quad \frac{\gamma_2}{\delta_2} = \frac{c + c}{d + d} = 2\frac{d - d}{c - c} \dots (51b),$$

$$\frac{\eta_1}{\phi_1} = \frac{e + e}{f - f} = 2\frac{f + f}{e - e}, \quad \frac{\eta_2}{\phi_2} = \frac{e + e}{f + f} = 2\frac{f - f}{e - e} \dots (51c),$$

$$\frac{\eta_1'}{\phi_1'} = \frac{e' + e'}{f' - f'} = 2\frac{f' + f'}{e' - e'}, \quad \frac{\eta_2'}{\phi_2'} = \frac{e' + e'}{f' + f'} = 2\frac{f' - f'}{e' - e'} \dots (51d),$$

* These are Euler's Rules.

and, since the two " 2^{10} parts" of each of the partitions ($50a, b$) are always mutually prime, it follows that, when the above fractions have been *reduced to their lowest terms*, then

The reduced numerators give $\alpha_1, \alpha_2; \gamma_1, \gamma_2; \&c. \dots\dots\dots(52a).$

The reduced denominators give $\beta_1, \beta_2; \delta_1, \delta_2; \&c. \dots\dots\dots(52b).$

The formulæ ($51a-d$) are liable to variation owing to the double signs ($\pm$) of the " 2^{10} parts" in the formulæ ($47a, b$) for N or $\frac{1}{2}N$:

[As the signs are independent, this gives four variations for each of the formulæ ($51a-d$), and is liable to cause some trouble in their use, as the correct sign can (as far as is yet known) only be found by trial].

27. Dimorph Quartans. One of the most interesting instances of *dimorphism* is that wherein the 4-tan form of N is itself dimorph, so that

$$N = x^4 + y^4 = x'^4 + y'^4 \dots\dots\dots(54).$$

Here each of these 4-tan forms has the four (algebraic) 2^{10} partitions of Art. 4: each of the latter forms is therefore a 2^{10} *dimorph*.

Euler has treated of the mode of constructing these numbers in three* *Memoirs* (1772-1780), and in the 2nd *Memoir* gives† explicit general formulæ for the sums and differences ($x \pm x'$), ($y \pm y'$) from which those for x, y, x', y' can be at once written down. M. Desboves has also treated‡ this question (1880) independently, and has given general formulæ for x, y, x', y' which are identical with those drawn as stated from Euler's results. These are—(in changed notation) —

$$x = \lambda \cdot \{(t^2 + u^2)(2t^5 - t^4u + 2t^3u^2 + 18t^2u^3 - u^5) + 8tu^2(2t^4 + u^4)\} \dots\dots\dots(55a),$$

$$y = \lambda \cdot \{(t^2 + u^2)(t^5 - 18t^3u^2 + 2t^2u^3 + tu^4 + 2u^5) + 8t^2u(t^4 + 2u^4)\} \dots\dots\dots(55b),$$

$$x' = \lambda \cdot \{(t^2 + u^2)(2t^5 + t^4u + 2t^3u^2 - 18t^2u^3 + u^5) + 8tu^2(2t^4 + u^4)\} \dots\dots\dots(55c),$$

$$y' = \lambda \cdot \{(t^2 + u^2)(-t^5 + 18t^3u^2 + 2t^2u^3 - tu^4 + 2u^5) + 8t^2u(t^4 + 2u^4)\} \dots\dots\dots(55d),$$

where t, u are any two (mutually prime) integers.....(56a),

$\lambda = 1$, if $tu = \epsilon$; $\lambda = \frac{1}{64}$, if $tu = w$ (56b).

* Euler's *Comment. Arithm.*, Petrop., 1819; t. i, pp. 473-476; t. ii, pp. 281-293; t. ii, pp. 450-456.

† *Op. cit.*, t. ii, pp. 238, 289. Euler's p, q, r, s are the present x, x', y, y' ; his f, g are the present t, u .

‡ Desboves's *Sur la résolution en nombres entiers ou complexes de l'Equation* $Un \pm V^n = S^n + W^n$, (Assoc. Franç. pour l'Avancement des Sciences, 1880). He quotes only Euler's first Memoir, and seems not to have seen Euler's production of these formulæ in 1778. His U, V, S, W are the present x, y, x', y' ; his x, y are the present t, u ; but his formula for W requires correction (as above). His first Example contains a misprint (4176 should be 1176).

These formulæ are of the 7th degree in t, u , so that x, y, x', y', N rise in magnitude very rapidly with t, u : they always yield one of x, y and one of x', y' even, so that N is always odd.

Euler's first* Memoir gives a different general method for constructing these numbers: the formulæ are quite general, and could be readily reduced to a form like (55a-d) giving the values of x, y, x', y' explicitly: but these formulæ would rise to about the 17th degree† in the parameter (Euler's b) involved, so that it does not seem worth while detailing them here. The Method, however, is of great interest, as it appears to lead *only to odd values* of x, y, x', y' , and therefore to even values of N , which do not arise in the formulæ (55a-d).

Euler's third Memoir‡ contains a special method (which it does not seem easy to generalise) of constructing a single example of these numbers, which does not seem to be deducible from either of the preceding general methods. A really exhaustive investigation of this difficult problem is a desideratum.

27a. *Dimorph Quartans, Factorisation:*

The factorisation is given by (49a) as

$$N = \frac{(xy')^4 - (yx')^4}{(y^4 - y'^4) = (x^4 - x'^4)} = \frac{(xx')^4 - (yy')^4}{(x'^4 - y'^4) = (x^4 - y'^4)} = L.M \dots\dots\dots (57).$$

These apparently break up into three§ factors each, for

$$N = \frac{(xy' - yx')(xy' + yx')(xy')^2 + (yx')^2}{(y' - y)(y' + y) \quad y'^2 + y^2} = \frac{(xx' - yy')(xx' + yy')(xx')^2 + (yy')^2}{(x' - y)(x' + y) \quad x'^2 + y'^2} \dots (58),$$

where the denominators may be changed respectively to

$$(x - x')(x + x')(x^2 + x'^2), \text{ or } (x - y')(x + y')(x^2 + y'^2) \dots\dots\dots (58a).$$

But two sets of L, M may also be readily formed from their (α, β) 2^{ic} partitions; for by (51a)

$$\text{One set has } \frac{\alpha_1}{\beta_1} = \frac{x'^2 + x^2}{y^2 - y'^2} = \frac{y^2 + y'^2}{x'^2 - x^2}; \quad \frac{\alpha_2}{\beta_2} = \frac{x'^2 + x^2}{y^2 + y'^2} = \frac{y^2 - y'^2}{x'^2 - x^2} \dots\dots\dots (59a),$$

* *Op. cit.*, t. i, pp. 473-476.

† This does not necessarily involve very high numbers in the result: as the x, y, x', y' directly given by the formulæ sometimes contain a high common factor which may be cancelled. *E.g.* Euler's second example, pp. 475, 476, is incorrectly worked (the last on p. 475—and along with it all that follows—is incorrect): the corrected values (given on t. ii, p. 456) are $\gg 12231$; the original incorrect figures are reprinted in M. Desboves's Memoir (above quoted) with a new misprint.

‡ *Op. cit.*, t. ii, pp. 450-456.

§ It is, however, possible that one of the three factors of the numerator might cancel out, thus leaving only two factors in the numerator: but see Art. 27b.

And one set has $\frac{\alpha_1'}{\beta_1'} = \frac{x'^2 + y'^2}{x'^2 - y'^2} = \frac{x^2 + y'^2}{x'^2 - y'^2}$; $\frac{\alpha_2'}{\beta_2'} = \frac{x'^2 + y'^2}{x'^2 + y'^2} = \frac{x^2 - y'^2}{x'^2 - y'^2} \dots\dots (59b).$

Here $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ are given by the Rules (52*a, b*) ; and L, M are then given by (50*a, b*).

As N has thus been resolved into co-factors (L, M) in two different ways, whether by (58) or by (59*a, b*), it is clear that L, M may be usually further resolved thereby (into two co-factors each).

27*b. Odd Dimorphs, Algebraical Factorisation.* In the case of odd Dimorph Quartans ($xy = \epsilon, x'y' = \epsilon, N = \Omega$) ; where x, y, x', y' are given by (55*a-d*), the late Mr. C. E. Bickmore discovered* the algebraic resolution of each of L, M into two co-factors, and therefore of N into four co-factors, say

$$L = L'.L'', M = M'.M''; N = LM = (L'L'')(M'M'') \dots\dots\dots (60),$$

and has presented them in their (a, b), (c, d) partitions, as follows*

$$L' = \lambda'. \{t^4 + u^4\} \dots\dots\dots (61a),$$

$$L'' = \lambda''. \{(t^4 + 14t^2u^2 + u^4)^2 + 4^2(t^4 - u^4)^2\} \dots\dots\dots (61b),$$

$$= \lambda'''. \{(3t^4 + 10t^2u^2 + 3u^4)^2 + 2(2t^4 - 4t^2u^2 + 2u^4)^2\} \dots\dots\dots (61b'),$$

$$M' = \mu'. \{(t^4 - 10t^2u^2 + u^4)^2 + (4tu)^2(t^2 + u^2)^2\} \dots\dots\dots (61c);$$

$$= \mu''. \{(t^4 - 2t^2u^2 + u^4)^2 + 2(8t^2u^2)^2\} \dots\dots\dots (61c'),$$

$$M'' = \mu''. \{(t^4 - 2t^2u^2 + u^4)^2 + (8tu)^2(t^2 + u^2)^2\} \dots\dots\dots (61d),$$

$$= \mu'''. \{(t^4 + 14t^2u^2 + u^4)^2 + 2(4tu)^2(t^2 - u^2)^2\} \dots\dots\dots (61d'),$$

where $\lambda' = \lambda'' = \mu' = \mu'' = 1$, when $tu = \epsilon \dots\dots\dots (62a);$

$$\lambda' = \frac{1}{2}, \lambda'' = \frac{1}{2\sqrt{5}}, \mu' = \frac{1}{2\sqrt{5}}, \mu'' = \frac{1}{2\sqrt{5}}, \text{ when } tu = \omega \dots\dots\dots (62b).$$

This algebraic resolution of these Dimorph Quartans into four co-factors has rendered possible the *complete factorisation* (into prime factors) of extremely high numbers of this kind.

The following remarkable relation holds between the Residues of $L'', M', M'' \pmod{L'}$,

$$L'' \equiv M' \equiv M'' \equiv 132t^4u^4 \pmod{L'}, \text{ when } tu = \epsilon \dots\dots\dots (63a),$$

$$L''/\lambda'' \equiv M'/\mu' \equiv M''/\mu'' \equiv 132t^4u^4 \pmod{L'/\lambda'}, \text{ when } tu = \omega \dots\dots\dots (63b).$$

27*c. Odd Dimorphs, Complex Factors.* The late Mr.

* Communicated to the author in M.S. in 1893: it is believed that they have not been hitherto published.

Bickmore discovered* also the complex 4-tic factors of the above four algebraic factors L', L'', M', M'' ; viz.:

If ρ be an imaginary root of $\rho^4 + 1 \equiv 0$,

and $N\phi(\rho)$ denotes "Norm of $\phi(\rho)$,"

where $\phi(\rho)$ denotes a 4-tic complex number; then—

$$L' = N(t + u\rho) \dots\dots\dots (64a),$$

$$L'' = N\{(t^2 - u^2) + 2(t^2 + u^2)\rho + 4(t^2 + u^2)\rho^2\} \dots\dots\dots (64b),$$

$$M' = N\{(t^2 - u^2) + 2tu(\rho + \rho^2)\} \dots\dots\dots (64c),$$

$$M'' = N\{(t + u)^2 + (t^2 - u^2)\rho + (t - u)^2\rho^2\} \dots\dots\dots (64d).$$

27d. *Dimorph Quartans, Examples.* The Table below gives a number of examples† of these numbers.

Ex. 1 to 13 are examples of formulæ (55a-d), with the four algebraic factors L', L'', M', M'' of formulæ (61a-d') completely factorised. The whole of the cases within the powers of the large Factor-Tables are here given.

Ex. 14, 15 are Euler's own examples of his first method.

Ex. 16 is Euler's own example of his third method.

Ex	t, u	x	y	x'	y'	L'	L''	M'	M''
1	2, 1	1203,	76	653,	1176	17;	8929;	2129;	6481;
2	2, 3	40465,	11888	2513,	40540	97;	428801;	73.2281;	390001;
3	4, 1	53935,	31494	52881,	35710	257;	521.2441;	89.937;	346561;
4	4, 3	419909,	31238	81659,	419762	337;	6026609;	257.10337;	73.193.409;
5	2, 5	444311,	345588	108201,	480032	641;	137.73721;	353.4177;	5576881;
6	1, 3	134,	133	59,	158	41;	569;	113;	241;
7	1, 5	1623,	3494	2338,	3351	313;	28057;	3217;	5521;
8	3, 5	17236,	6673	529,	17332	353;	73.1049;	51137;	97.673;
9	1, 7	6484,	32187	23109,	29812	1201;	41.9689;	73.601;	51361;
10	3, 7	84545,	59678	15322,	89345	17.73;	629081;	214481;	380881;
11	5, 7	287178,	67429	20773,	287394	17.89;	401.4457;	1049.1433;	1678321;
12	1, 9	7805,	174484	125516,	161405	17.193;	97.30113;	326561;	73.4057
13	5, 9	875539,	400262	3106,	884947	3593;	137.52081;	233.17929;	337.16993
14	?	103,	542	359,	514		17;	63113;	80537
15	b=3	2903,	12231	10203,	10381		2.82361;	17.193.577.71993	
16	b=2	2219449,	555617	1584749,	2061283	‡		‡	

27e. *Dimorphs with algebraic factors.* Writing, as in (54), (60),

$$N = x^4 + y^4 = x'^4 + y'^4 = L' L'' . M' M'' \dots\dots\dots (65),$$

* These were communicated to the author in 1898: it is believed that they have not been published hitherto. It is beyond the scope of this Paper to deal with the complex theory of Quartans; but it is thought worth while to place these remarkable results on record.

† This work has been verified by Miss B. E. Haselden.

‡ The composition of these large numbers is not known.

whereof, by (61a), the smallest factor $L' = \lambda'(t^4 + u^4)$ is—except for the coefficient λ' —of the same form as N , and, using ϕ, ψ, ϕ', ψ' to denote the functions (55a-d), so that

$$x = \phi(t, u), y = \psi(t, u); \quad x' = \phi'(t, u), y' = \psi'(t, u) \dots (66a),$$

and taking t, u, t', u' the similar functions of new auxiliaries (t_1, u_1) , so that

$$t = \phi(t_1, u_1), u = \psi(t_1, u_1); \quad t' = \phi'(t_1, u_1), u' = \psi'(t_1, u_1) \dots (66b),$$

then, by (61a) with Art. 27

$$L' = \lambda(t^4 + u^4) = \lambda'(t'^4 + u'^4) \dots \dots \dots (67a).$$

Hence, by Art. 27b, L' itself is (algebraically) resolvable into four factors—disregarding the coefficient λ' —say

$$L' = \lambda'(L'_1 L''_1 M'_1 M''_1) \dots \dots \dots (67b),$$

whence

$$N = \lambda'(L'_1 L''_1 M'_1 M''_1) \cdot L'' M' M'' \dots (68),$$

so that N is hereby resolvable (algebraically) into *seven* factors.

And, since by (61a) the smallest factor $L'_1 = \lambda'_1(t_1^4 + u_1^4)$ is of the same form as the original N , this process may be repeated again, with similar results, and may therefore be repeated *ad inf.* Thus Dimorph Quartans may be produced containing any desired number of algebraic factors.

[Unfortunately, this process only applies to the *smallest* algebraic factors L, L'_1, L'_2 , &c., at each step. The larger factors L'', M', M'' at each step are always far beyond the power of the large Factor-Tables, so that the complete factorisation of these large Dimorphs N is at present practically unattainable.]

27f. Allied Quartic Dimorphs. The values of x, y, x', y' which yield Dimorph Quartans give rise obviously to two solutions each of the following* 4-tic forms

$$1^\circ. \quad x^4 - x'^4 = y^4 - y'^4; \quad \& \quad 2^\circ. \quad x^4 - y'^4 = x'^4 - y^4 \dots \dots \dots (69a),$$

$$\text{or} \quad 1^\circ. \quad (x^2 - x'^2)(x^2 + x'^2) = (y'^2 - y^2)(y'^2 + y^2) \quad \} \dots \dots \dots (69b),$$

$$\text{and} \quad 2^\circ. \quad (x^2 - y'^2)(x^2 + y'^2) = (x'^2 - y^2)(x'^2 + y^2) \quad \}$$

Also, writing

$$1^\circ. \quad x = \xi + \xi', \quad x' = \xi - \xi', \quad y' = \eta' + \eta, \quad y = \eta' - \eta \dots \dots \dots (70a),$$

$$\text{or} \quad 2^\circ. \quad x = \xi + \eta', \quad y' = \xi - \eta', \quad x' = \xi' + \eta, \quad y = \xi' - \eta \dots \dots \dots (70b),$$

the Results (55a-d) yield two solutions of the dimorphs

$$1^\circ. \quad \xi \xi' (\xi^2 + \xi'^2) = \eta \eta' (\eta^2 + \eta'^2); \quad 2^\circ. \quad \xi \eta' (\xi^2 + \eta'^2) = \xi' \eta (\xi'^2 + \eta^2) \dots \dots \dots (71).$$

* This solution is given by Euler in connexion with that of the Dimorph-Quartan. *Op. cit.*, t. ii, pp. 282, 287-293.

27g. *Geometrical Illustration.* Let (a, b, c) , (a', b', c') be the sides— c, c' being the hypotenuses—of two right-angled triangles. Then a, b, c , &c., have the forms

$$c = t^2 + u^2, a = t^2 - u^2, b = 2tu; c' = t'^2 + u'^2, a' = t'^2 - u'^2, b' = 2t'u' \dots (72),$$

and the above Results (69a), (71) evidently give two solutions for each set of (x, y, x', y') , and of (ξ, η, ξ', η') of the two following geometric problems, viz.

To find a, b, c, a', b', c' (all integers) so that

$$1^\circ. ac = a'c'; \quad 2^\circ. bc = b'c' \dots \dots \dots (73).$$

28. *Impossible Forms.* It is known* that neither a Quartan, nor a Half-Quartan, can be a Square. The question arises whether either of them can be a *Bin-Aurifeuilliant*† $(x^4 + 4y^4)$.

If possible, let—

$$N = x^4 + y^4 = \xi^4 + 4\eta^4 = (\xi^2 - 2\eta^2)^2 + (2\xi\eta)^2.$$

Here, x being prime to y ; x odd, y even (Art. 1a), involve

$$\xi \text{ prime to } \eta, \text{ and } \xi \text{ odd.}$$

And, since $y^4 \equiv 4\eta^4$, this involves

$$x^2 = \xi^2 - 2\eta^2, y^2 = 2\xi\eta; \text{ whence } \xi = t^2, \eta = 2u^2.$$

These involve $t^4 - 2u^4 = x^2$ (a square),

and this is known ‡ to be impossible.

Again, if possible, let—

$$\frac{1}{2}N = \frac{1}{2}(x^4 + y^4) = \xi^4 + 4\eta^4.$$

Here, x being prime to y , and x, y both odd, (Art. 1a) involve

$$\xi \text{ prime to } \eta, \text{ and } \xi \text{ odd.}$$

$$\text{Hence } \left(\frac{x^2 + y^2}{2}\right)^2 = \left(\frac{x^2 - y^2}{2}\right)^2 = \xi^4 + 4\eta^4 = (\xi^2 - 2\eta^2)^2 + (2\xi\eta)^2.$$

$$\text{Hence, } 1^\circ. x^2 + y^2 = 2\xi^2 \text{ and } x^2 - y^2 = (2\eta)^2,$$

$$\text{which involve } \xi^2 + 2\eta^2 = x^2, \xi^2 - (2\eta)^2 = y^2,$$

which are known § to be impossible together,

$$\text{or, } 2^\circ. x^2 + y^2 = 2\xi^2 - 4\eta^4, \text{ and } x^2 - y^2 = 4\xi\eta,$$

* Euler's *Elements of Algebra*, Hewlett's transl., Lond., 1822, pp. 406–410.

† This name is given to the quantity $(x^4 + 4y^4)$ in the author's Paper *On Aurifeuillians*, in *Proc. Lond. Math. Soc.*, Vol. XXI., pp. 381–438, where its properties are studied. The chief property—(connected with factorisation)—is that it is algebraically resolvable into two factors.

‡ see M. LeBesgne's Paper *Résolution des Equations Biquadratiques*, in *Journ. des Math. Pures et Appl.*, t. xviii, p. 83, &c.

§ see Euler's *Algebra*, p. 410, shewing $x^4 - 4y^4 \neq \square$.

which (since xy is odd) involves $x \pm y = 2\xi$, $x \mp y = 2\eta$,
whence $x^2 + y^2 = 4\xi^2 + 4\eta^2$; and $= 2\xi^2 - 4\eta^2$ (by above),
an impossible Result,

Hence,

“Quartans, and Half-Quartans, cannot be Bin-Aurifeuillians”....(73),

[This removes one chance of factorisation].

29. Product Quartans. The question arises whether a composite Quartan (N) or Half-Quartan ($\frac{1}{2}N$) can be a product of two or more Quartans, or Half-Quartans, or of both. As no general Rules are known for forming these numbers, it must suffice to give a few examples, showing that the case is fairly common.

The following abbreviations are used—for brevity—

(x, y) denotes $(x^4 + y^4)$, [x odd, y even].....(74a),

$\{x, y\}$ denotes $\frac{1}{2}(x^4 + y^4)$, [xy odd](74b).

N or $\frac{1}{2}N$	Factors.	4-tan factors.
$(13, 8) = 17$; 17.113;		$= (1, 2) \cdot (5, 6)$
$(23, 14) = 97$; 17.193;		$= (3, 2) \cdot \{1, 9\}$
$(15, 4) = 41$; 17.73 ;		$= \{1, 3\} \cdot \{7, 3\}$
$\{17, 33\} = 41$; 113.137;		$= \{1, 3\} \cdot \{13, 7\}$
$(13, 2) = 17$; 41; 41;		$= (1, 2) \cdot \{1, 3\} \cdot \{1, 3\}$
$(37, 24) = 17$; 17; 17.449;		$= (1, 2) \cdot (1, 2) \cdot \{11, 5\}$
$(129, 2) = 41$; 41; 257; 641;		$= \{1, 3\} \cdot \{1, 3\} \cdot (1, 4) \cdot (5, 2)$

Hereby the following forms are seen to be possible

$(X, Y) = \text{Any of } (x_1, y_1)(x_2, y_2), (x_1, y_1)\{x_2, y_2\}, \{x_1, y_1\}\{x_2, y_2\} \dots (74c),$

$\{X, Y\} = \{x_1, y_1\} \cdot \{x_2, y_2\} \dots (74d).$

30. Quartan Nexus. Using the symbol N' to denote either of N or $\frac{1}{2}N$, a system of factorisable Quartans or Half-Quartans

$$N'_1 = L_1 M_1, \quad N'_2 = L_2 M_2, \quad \dots, \quad N'_r = L_r M_r \dots (75),$$

so related that

$$M_1 = L_2, \quad M_2 = L_3, \quad \dots, \quad M_{r-1} = L_r; \quad \text{lastly } M_r = L_1 \dots (76),$$

may be styled a *Quartan Nexus*. Rules for their formation, and Examples, will be found in Art. 33b.

When the number (r) of members (N'_r) in a Nexus is *even*, it follows at once that

$$N'_1 N'_3 N'_5 \dots N'_{2r-1} = N'_2 N'_4 N'_6 \dots N'_{2r} \dots (77),$$

31. Square Quartan Products. These may be formed in several ways, which will be obvious from Art. 29, 30.

1°. Every Product-Quartan (Art. 29) multiplied by its Quartan-factors gives a perfect square product(78a).

2°. Every Quartan Nexus (Art. 30) has the continued product of all its members = a square(78b).

More generally, using each of the symbols N' , N'' to denote either Quartans or Half-Quartans, as required, let

$$N_1'' = m_1^2 N_1', N_2'' = m_2^2 N_2', \dots, N_r'' = m_r^2 N_r', [\text{any } m_r \text{ may} = 1] \dots (79),$$

then $N_1'' N_2'' \dots N_r'' = (m_1 m_2 \dots m_r)^2 \cdot (N_1' N_2' \dots N_r')$

$$= \square, \text{ if } N_1', N_2' \dots N_r' = \square \dots (80),$$

[i.e. if $N_1', N_2', \&c., N_r'$ are a 4-tan Nexus].

No examples of 1°, 2° are needed: the following are examples of the more general case. Notation as in Results (74a, b).

$$1^\circ. (1, 2) = 1.17, (13, 2) = 17.41^2; \therefore (1, 2)(13, 2) = \square.$$

$$2^\circ. (13, 8) = 17^2.113, (11, 32) = 97^2.113; \therefore (13, 8)(11, 32) = \square.$$

$$3^\circ. (13, 8) = 17^2.113, (13, 7) = 113.2.137; (15, 13) = 2.137.17^2; \therefore \text{Product} = \square.$$

[For another method see Art. 61].

32. Form $(X^4 - 4Y^4)$. If possible let

$$N = x^4 + y^4 = X^4 - 4Y^4 = L.M \dots (81),$$

and let

$$L = \alpha_1^2 + \beta_1^2 = \gamma_1^2 + 2\delta_1^2 = \epsilon_1^2 - 2\phi_1^2 \dots (81a),$$

$$M = \alpha_2^2 + \beta_2^2 = \gamma_2^2 + 2\delta_2^2 = \epsilon_2^2 - 2\phi_2^2 \dots (81b).$$

$$\text{Then } L = X^2 - 2Y^2, \quad M = X^2 + 2Y^2 \dots (81c).$$

$$\text{These involve } X = \epsilon_1 = \gamma_2, \quad Y = \phi_1 = \delta_2 \dots (81d),$$

$$\text{i.e. The } \epsilon, \phi \text{ of } L \text{ are the same as the } \gamma, \delta \text{ of } M \dots (81e).$$

$$\text{Also } X^4 - x^4 = y^4 + 4Y^4, \quad \& \quad X^4 - y^4 = x^4 + 4Y^4 \dots (81f).$$

Hence every solution (x, y, X, Y) of (81) gives two Bin-Aurifeuillians (algebraically) resolvable into three factors, viz.

$$(X-x)(X+x)(X^2+x^2) = (y^4+4Y^4) \dots (82a),$$

$$(X-y)(X+y)(X^2+y^2) = (x^4+4Y^4) \dots (82b).$$

It does not seem obvious how to solve them in any general manner; but, supposing

$$x, X \text{ both odd; } y, Y \text{ both even,}$$

a simple case is obtained by

$$X = m + 1, \quad x = m - 1; \text{ so that } X - x = 2, \quad X + x = 2m \dots\dots\dots(83),$$

whereby Eq. (82a) becomes

$$8m(m^2 + 1) = y^4 + 4Y^4 \dots\dots\dots(84),$$

of which two simple solutions are given by

$$1^\circ. \text{ Take } m = \frac{1}{2}\eta^4, \quad y/Y = \eta^2, \quad Y = \eta, \text{ whence } y = \eta^3 \dots\dots\dots(85a),$$

which reduces both sides of (84) to $\eta^4(\eta^8 + 4)$.

$$2^\circ. \text{ Take } m = 2\eta^4, \quad Y/y = \eta^2, \quad y = 2\eta \dots\dots\dots(85b),$$

which reduces both sides of (84) to $(2\eta)^4(4\eta^8 + 1)$.

The two simple forms of N hence resulting, together with the three 2^{ic} partitions of their algebraic factors (L, M), are shown in the schemes below

$$1^\circ. \quad N = (\frac{1}{2}\eta^4 - 1)^4 + (\eta^3)^4 = L.M \dots\dots\dots(86),$$

$$L = (\eta^2 - 1)^2 + (\frac{1}{2}\eta^4)^2 \qquad M = (\eta^2 + 1)^2 + (\frac{1}{2}\eta^4)^2 \dots\dots\dots(86a),$$

$$= (\frac{1}{2}\eta^4 - 2\eta^2 + 1)^2 + 2(\eta^3 - \eta)^2 \qquad = (\frac{1}{2}\eta^4 + 1)^2 + 2\eta^2 \dots\dots\dots(86b),$$

$$= (\frac{1}{2}\eta^4 + 1)^2 - 2\eta^2 \qquad = (\frac{1}{2}\eta^4 + 2\eta^2 + 1)^2 - 2(\eta^3 + \eta)^2 \dots\dots\dots(86c).$$

$$2^\circ. \quad N = (2\eta^4 - 1)^4 + (2\eta)^4 = L.M \dots\dots\dots(87),$$

$$L = 1^2 + [2\eta^2(\eta^2 - 1)]^2 \qquad M = 1^2 + [2\eta^2(\eta^2 + 1)]^2 \dots\dots\dots(87a),$$

$$= (2\eta^4 - 4\eta^2 + 1)^2 + 2[2\eta(\eta^2 - 1)]^2 \qquad = (2\eta^4 + 1)^2 + 2(2\eta^3)^2 \dots\dots\dots(87b),$$

$$= (2\eta^4 + 1)^2 - 2(2\eta^3)^2 \qquad = (2\eta^4 + 4\eta^2 + 1)^2 + 2[2\eta(\eta^2 + 1)]^2 \dots\dots\dots(87c).$$

The close relationship of the "parts" of the 2^{ic} partitions of the L, M is very remarkable, thus—

$$\text{Case } 1^\circ. \quad \beta_1 = \beta_2; \quad \text{Case } 2^\circ. \quad \alpha_1 = \alpha_2 \dots\dots\dots(88a),$$

$$\text{and in Both Cases } \epsilon_1 = \gamma_2, \quad \phi_1 = \delta_2 \dots\dots\dots(88b).$$

32a. Exceptional Case. When X, Y are such that $X^2 - 2Y^2 = +1$, then

$$L = 1, \quad N = LM = M = X^2 + 2Y^2,$$

and the factorisation fails, and N may in this case be prime; but only one such case is known, viz.

$$\text{Ex. } 17 = 1^4 + 2^4 = 9^4 - 4.2^4 = (3^2 - 2.2^2)(3^2 + 2.2^2) = 1.17;$$

$$32b. \text{ Conjugate form } (4Y^4 - X^4).$$

It suffices to note that this form cannot yield Quartans.

32c. General Formulæ. The above formulæ for N and for its factors L, M may be generalised by writing η/ξ for η ,

and clearing of fractions, so as to render the expressions homogeneous in ξ, η .

$$1^{\circ}. N = (\frac{1}{2}\eta^4 - \xi^4) + (\xi\eta^2)^4 = (\frac{1}{2}\eta^4 + \xi^4) - 4(\xi^2\eta)^4 \dots\dots\dots (89a),$$

$$2^{\circ}. N = (2\eta^4 - \xi^4) + (2\xi^2\eta)^4 = (2\eta^4 + \xi^4) - 4(2\xi\eta^3)^4 \dots\dots\dots (89b).$$

Examples. Some of the most interesting examples arise by taking $\eta = 2^k$.

η	x, y	Form 1°.		x, y	Form 2°.	
		L	M		L	M
2	7, 8	73;	89;	31, 4	577;	2601;
4	127, 64	17.977;	16673;	511, 8	17.13553;	295937;
8	2047, 512	1289.3257;	4198529;	8191, 16	65028097;	97.137.5209;
16	32767, 4096	1073806849*;	17.89.457.1553;			

33. Form $N = (\alpha_1^2 + \beta^2)(\alpha_2^2 + \beta^2)$. This form (with β the same in both factors) gives rise to numerous series of (arithmetically) factorisable Quartans.

$$\text{Let } N = x^4 + y^4 = L \cdot M \dots\dots\dots (90),$$

$$\text{where } L = \alpha_1^2 + \beta^2, \quad M = \alpha_2^2 + \beta^2 \dots\dots\dots (90a).$$

$$\text{Then } LM = (\alpha_1\alpha_2 \mp \beta^2)^2 + \{\alpha_1 \pm \alpha_2\}\beta^2 = (x^2)^2 + (y^2)^2,$$

$$\text{whence } x^2 = \beta^2 \mp \alpha_1\alpha_2, \quad y^2 = (\alpha_1 \pm \alpha_2)\beta.$$

$$\text{Let } \alpha_1 \pm \alpha_2 = my, \quad y = m\beta \dots\dots\dots (91a).$$

$$\text{Then } x^2 = \frac{1}{m^4}(\alpha_1 \pm \alpha_2)^2 \mp \alpha_1\alpha_2 = \frac{1}{m^4}\{\alpha_1^2 \mp (m^4 - 2)\alpha_1\alpha_2 + \alpha_2^2\} \dots\dots (91b).$$

This Diophantine equation can be solved (*i.e.* values of α_1, α_2 can be found which shall make the dexter a perfect square) for all values of m . Hence, every value of m yields an infinite series of (arithmetically) factorisable Quartans. In some cases the algebraical solution, and factorisation, can be effected.

33a. Simple case $m = 1$. The Diophantine (91b) becomes

$$\alpha_1^2 \pm \alpha_1\alpha_2 + \alpha_2^2 = x^2 \dots\dots\dots (92),$$

$$\text{whence, either } (\alpha_1 \pm \frac{1}{2}\alpha_2)^2 + 3(\frac{1}{2}\alpha_2)^2 = x^2, [\alpha_2 \text{ even}] \dots (92a),$$

$$\text{or } \{\frac{1}{2}(\alpha_1 \mp \alpha_2)\}^2 + 3\{\frac{1}{2}(\alpha_1 \pm \alpha_2)\}^2 = x^2, [\alpha_1 \& \alpha_2 \text{ odd}] \dots (92b).$$

Hence x may be any number of form

$$x = A^2 + 3B^2, \quad \text{whence } x^2 = A^2 + 3B^2 \dots\dots\dots (93),$$

$$\text{where } A = A^2 - 3B^2, \quad B = 2AB \dots\dots\dots (93a),$$

* The composition of this number is not known.

whence arise three solutions ($1^\circ, 2^\circ, 3^\circ$) distinguished below by accents,

$$1^\circ. \quad a_1' = A \sim B, \quad a_2' = 2B, \quad \beta' = a_1' + a_2' = A + B \dots\dots\dots(94a),$$

$$2^\circ. \quad a_1'' = A + B, \quad a_2'' = 2B, \quad \beta'' = a_1'' - a_2'' = A \sim B \dots\dots\dots(94b),$$

$$3^\circ. \quad a_1''' = A \sim B, \quad a_2''' = A + B, \quad \beta''' = a_1''' \sim a_2''' = 2B \dots\dots\dots(94c),$$

and, in each case $y = a_1 \pm a_2 = \beta \dots\dots\dots(94d).$

Case of $AB = \omega$. This gives each of a_1, a_2, β, x, y a multiple of 4: on cancelling this throughout, the same results ensue as are found (more simply) by taking $AB = \varepsilon$, as below.

Case of $AB = \varepsilon$. Taking A, B one odd, one even (but mutually prime); note that—

Formulæ $1^\circ, 2^\circ$ give x, y odd, N even.

Formulæ 3° give x odd, y even, N odd.

33b. Quartan Triads (N', N'', N'''). Taking $AB = \varepsilon$, the three Quartans (N', N'', N''') obtained from any one pair A, B by formulæ $1^\circ, 2^\circ, 3^\circ$ have a remarkable connexion.

Writing $N' = L'M', \quad N'' = L''M'', \quad N''' = L'''M''' \dots\dots\dots(95).$

Then $a_1' = \beta'' = a_1''', \quad a_2' = a_2'' = \beta''', \quad \beta' = a_1'' = a_2''' \dots\dots\dots(95a),$

$$x' = x'' = x''', \quad y''' = y' \pm y'' \dots\dots\dots(95b),$$

where $L' = a_2'^2 + \beta'^2, \quad L'' = a_1''^2 + \beta''^2, \quad L''' = a_1'''^2 + \beta'''^2 \dots\dots\dots(95c),$

$$M' = a_1'^2 + \beta'^2, \quad M'' = a_2''^2 + \beta''^2, \quad M''' = a_2'''^2 + \beta'''^2 \dots\dots\dots(95d),$$

whence $M' = L'', \quad M'' = L''', \quad M''' = L' \dots\dots\dots(95e),$

so that N', N'', N''' form a Quartan Nexus (Art. 30) $\dots\dots\dots(95f),$

and $N'N''N''' = (L'L''L''')^2 = (M'M''M''')^2 \dots\dots\dots(95g).$

Ex. The above results are all well shown in the five Triads in the Table below, each line of which shows the elements, *i.e.* the $(x, y), (a_1, a_2, \beta), L, M$ of (one of) the N', N'', N''' resulting from a single pair of A, B as data.

A, B	x, y	a_1, a_2, β	$L ; M$
2, 1	7, 5	-3, 8, 5	89; 2.17;
	7, 3	5, 8, -3	2.17; 73;
	7, 8	3, 5, 8	73; 89;
1, 2	13, 15	7, 8, 15	17.17; 2.137;
	13, 7	15, 8, 7	2.137; 113;
	13, 8	7, 15, 8	113; 17.17;
4, 1	19, 21	5, 16, 21	17.41; 2.233;
	19, 5	21, 16, 5	2.233; 281;
	19, 16	5, 21, 16	281; 17.41;
1, 4	49, 55	39, 16, 55	17.193; 2.2273;
	49, 39	55, 16, 39	2.2273; 1777;
	49, 16	39, 55, 16	1777; 17.193;
2, 3	31, 35	11, 24, 35	1801; 2.673;
	31, 11	35, 24, 11	2.673; 17.41;
	31, 24	11, 35, 24	17.41; 1801;

Writing (x, y) as an abbreviation for $(x^4 + y^4)$, the Quartan Triad may be written

$$N' = (x, y'), \quad N'' = (x, y''), \quad N''' = (x, y''') \dots (96),$$

where each has the same x . By taking x composite, so that x may be expressed in several ways in the form $(A^2 + 3B^2)$, each of these ways leads to a different Quartan Triad (all with this same x).

Ex. Take $x = 91 = 7 \cdot 13 = 8^2 + 3 \cdot 3^2 = 4^2 + 3 \cdot 5^2$ (two forms).

Then the two Triads are*,

$$\begin{array}{l} (x, y') = L' : M' ; \quad (x, y'') = L'' : M'' ; \quad (x, y''') = L''' : M''' ; \\ (91, 85) = 41.101 : 2.3673 ; \quad (91, 11) = 2.3673 : 9337 ; \quad (91, 96) = 9337 : 41.101 ; \\ (91, 99) = 17.953 : 2.5081 ; \quad (91, 19) = 2.5031 : 6761 ; \quad (91, 80) = 6761 : 17.953 ; \end{array}$$

33c. Form $(x^{4n} + y^4)$. Factorisable Quartans of forms

$$N = X^4 + y^4, \quad \text{where } X = x^n \dots (97),$$

can be constructed by the procedure of Art. 33b, by simply taking x any number of form $x = a^2 + 3b^2$. Then $X = x^n$, and $X^2 = x^{2n}$ may be expressed in the forms

$$X = A^2 + 3B^2, \quad X^2 = A^2 + 3B^2,$$

by known rules. The formulæ of Art. 33a may now be used substituting X for x therein. Quartan Triads may thus be produced with the same $X = x^n$ in each number (N', N'', N''') ,

Ex. To form $N = x^8 + y^4$.

Take $x = 7 = 2^2 + 3 \cdot 1^2$; then $X = x^2 = 1^2 + 3 \cdot 4^2$.

This example forms the 4th Triad of the Table of Art 33b, *q.v.* The results are the Triad

$$N' = 7^8 + 55^4; \quad N'' = 7^8 + 39^4; \quad N''' = 7^8 + 16^4;$$

Here $N''' = 7^8 + 4^8$, an 8-*van*; but this is accidental, as the rules given do not readily provide y a square.

33d. Simple Case $m = 2$. The Diophantine becomes

$$\frac{1}{16}(\alpha_1^2 \mp 14\alpha_1\alpha_2 + \alpha_2^2) = x^2 \dots (98),$$

$$\text{or} \quad \alpha^2 - 3\alpha_2^2 = x^2, \quad \text{writing } \alpha = \frac{1}{2}(\alpha_1 \mp 7\alpha_2) \dots (98a),$$

This case will be found fully developed in Art. 52.

34. *Connexion with Pythagoreans.* If X, Y, Z be a set of Pythagorean numbers, *i.e.* such that

$$Z^2 = X^2 + Y^2 \dots (99),$$

* The colon (:) is here used to separate the co-factors $L \ M$ in each Quartan.

then they are of the following general type

$$X = x^2 - y^2, \quad Y = 2xy, \quad Z = x^2 + y^2 \dots\dots\dots(99a),$$

where x, y are any two, mutually prime, integers (x odd, y even).

Hence, writing—

$$x' = x - y, \quad y' = x + y \dots\dots\dots(100),$$

$$N = x^4 + y^4, \quad N' = x'^4 + y'^4 \dots\dots\dots(100a),$$

the Quartan and Pythagorean numbers are seen to be connected by the relations

$$N = \frac{1}{2}(X^2 + Z^2), \quad \frac{1}{2}N' = Y^2 + Z^2 = x^4 + 6x^2y^2 + y^4 \dots\dots\dots(100b),$$

$$2N + \frac{1}{2}N' = 3Z^2, \quad 2N - \frac{1}{2}N' = X^2 - Y^2 \dots\dots\dots(100c).$$

Hence the whole series of Quartans and Half-Quartans exist in conjugate pairs $(N, \frac{1}{2}N')$ along with an associated set of Pythagoreans (X, Y, Z) .

The (algebraic) 2^{ic} partitions $(a, b), (c, d), (e, f), (f', e')$ of these conjugate $N, \frac{1}{2}N'$ can be conveniently expressed in terms of X, Y, Z .

$$N = \{\frac{1}{2}(Z-X)\}^2 + \{\frac{1}{2}(Z+X)\}^2 = X^2 + 2(\frac{1}{2}Y)^2 = Z^2 - 2(\frac{1}{2}Y)^2 = 2(Z - \frac{1}{2}Y)^2 - (Z-Y)^2 \dots\dots\dots(101a),$$

$$\frac{1}{2}N' = Z^2 + Y^2 = X^2 + 2Y^2 = (2Z-X)^2 - 2(Z-X)^2 = 2Z^2 - X^2 \dots\dots\dots(101b).$$

Ex. Here follow some examples of the lowest of such associated numbers.

X	Y	Z	x	y	x'	y'	N	$\frac{1}{2}N'$
3,	4,	5	1,	2,	1,	3	17,	41
5,	12,	13	3,	2,	1,	5	97,	313
15,	8,	17	1,	4,	3,	5	257,	353
21,	20,	29	5,	2,	3,	7	641,	1773

35. *Quasi Half-Quartans.* The following form, obtained by writing $y\sqrt{2}$ for y in the formulæ (100a, b) for N' .

$$\frac{1}{2}N = \frac{1}{2}\{(x-y\sqrt{2})^4 + (x+y\sqrt{2})^4\} = x^4 + 12x^2y^2 + 4y^4 \dots\dots\dots(102),$$

is—from its being (algebraically) of the Half-Quartan form—closely allied to Half-Quartans, being of the same linear form $\frac{1}{2}N = 16n + 1$, and being also (algebraically) expressible in the same 2^{ic} forms

$$\begin{aligned} \frac{1}{2}N &= (x^2 - 2y^2)^2 + (4xy)^2 = (x^2 + 2y^2)^2 + 2(2xy)^2 \\ &= (x^2 + 6y^2)^2 - 2(4y^2)^2 = 2(x^2 + 2y^2)^2 - (x^2 - 2y^2)^2 \dots\dots\dots(102a), \end{aligned}$$

and has therefore all such properties as depend solely on the 2^{ic} forms closely similar to those of a Half-Quartan, including those of modularity (when $\frac{1}{2}N=p$) towards small bases (Art. 17-19). Also its divisors have the same linear and 2^{ic} forms as those of Quartans (Art. 5, 5a).

36. Mixed Quasi- and Real Half-Quartans. The form

$$H_n = \frac{1}{2}[(2^{\frac{1}{2}n} - 1)^2 + (2^{\frac{1}{2}n} + 1)^2] = 2^{2n} + 6 \cdot 2^n + 1 \dots\dots\dots(103),$$

which is a particular form of the N' of Results (100b), is interesting as including both Quasi- and Real Half-Quartans, according as n is *odd* or *even*. The linear and 2^{ic} forms are shown below—

Quasi Half-Quartans [n odd].

Real Half-Quartans [n even].

$$H_n = 16n + 1, [n \leq 1]$$

$$H_n = 16n + 1, [n > 2] \dots\dots(103a),$$

$$= (2^n - 1)^2 + (2^{\frac{1}{2}(n+3)})^2$$

$$= (2^n + 1)^2 + (2^{\frac{1}{2}(n+1)})^2 \dots\dots(103b),$$

$$= (2^n + 1)^2 + 2(2^{\frac{1}{2}(n+1)})^2$$

$$= (2^n - 1)^2 + 2(2^{\frac{1}{2}(n+1)})^2 \dots\dots(103c),$$

and both kinds have the *same* (e, f), and the same (f', e') partitions, *viz.*

$$H_n = (2^n + 3)^2 - 2 \cdot 2^2 = 2(2^n + 1)^2 - (2^n - 1)^2 \dots\dots\dots(102d),$$

and, when $H_n = p$, their modularity towards the bases ± 2 is given by the simple rules below

$H_n = p$	$(2/p)_2$	$(2/p)_4$	$(2/p)_8$	$(2/p)_{16}$	$(2/p)_{32}$
	+1	-1, +1	-1, +1	-1, +1	∓ 1
n (odd) = ω	1, 3	3, 5	5, 7	≤ 7	
n (even) = ϵ	2, 4	4, 6	6, 8	≤ 8	

The forms of the prime divisors (p) of H_n can be pretty closely defined.

Of course

$$p = 8\omega + 1 = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 \dots\dots\dots(104).$$

Now let Z, Z' be the two roots (each $< p$), of the congruence

$$Z^2 + 6Z + 1 = (Z + 3)^2 - 2 \cdot 2^2 \equiv 0 \pmod{p}.$$

And, noting that $p = e^2 - 2f^2$ gives $(e/f)^2 \equiv 2 \pmod{p}$,

so that $(Z + 3)^2 \equiv (2e/f)^2 \pmod{p}$,

whence $Z \equiv -3 + 2e/f, \quad Z' \equiv -3 - 2e/f \pmod{p} \dots\dots(105),$

$$Z + Z' \equiv -6, \quad ZZ' \equiv +1 \pmod{p} \dots\dots\dots(105a),$$

Hereby Z, Z' are both always possible; (the apparent fraction $2e/f$ is supposed to be reduced to its *integer value* for mod p).

Now, if $2^f \equiv +1 \pmod{p}$,

$$\text{and } \nu\xi = p-1 \text{ } [\xi \text{ a min., } \nu \text{ a max.}] \dots (106),$$

and, if n_0, n'_0 be the least indices (each $< \xi$) of the roots ($2^n, 2^{n'}$) of the congruence

$$H_n = 2^{2^n} + 6 \cdot 2^n + 1 \equiv 0 \pmod{p}$$

this involves $2^{n_0} \equiv Z, 2^{n'_0} \equiv Z' \pmod{p} \dots \dots (107),$

where $2^{n_0+n'_0} = 2^{n_0} \cdot 2^{n'_0} \equiv ZZ' \equiv +1 \pmod{p} \dots \dots (107a).$

Hence $n_0 + n'_0 = \xi \dots \dots \dots (107b),$

and the general values of n, n' , for which p is a divisor, are

$$n = m\xi \pm n_0, n' = m'\xi \mp n'_0 \dots \dots \dots (107c).$$

Hence n_0, n'_0 are both possible, or both impossible, together: and the condition of possibility is

$$(Z/p)_\nu = (Z'/p)_\nu = (2/p)_\nu = +1 \dots \dots \dots (108).$$

This condition—though sometimes helpful—is too difficult for general use: it may be shown to be identically satisfied when $\nu=2$, or 4. The results (107b, c) show that the divisors p belong to the odd or even kinds of H_n according to the following scheme:—

$$\begin{array}{l} \xi = \omega \text{ gives } n_0, n'_0 \text{ one odd, one even,} \\ \text{and } n \text{ alternates odd and even} \end{array} \left| \begin{array}{l} \xi = \varepsilon \text{ with } n_0 = \omega \text{ gives every } n \text{ odd} \\ \dots (109a), \\ \xi = \varepsilon \text{ with } n_0 = \varepsilon \text{ gives every } n \text{ even} \\ \dots (110b). \end{array} \right.$$

The following Table shews the *complete* factorisation* of the numbers H_n up to $n = 17$:—

n	1	3	5	7	9	11	13	15	17
H_n	17; 113; 1217; 17. 1009; 17. 15601; 4206593; 193. 347969; 17. 63172849; 17. 97. 10418833;								
n	2	4	6	8	10	12	14	16	18
H_n	41 353; 4481; 67073; 1054721; 16801793; 8009. 33529; 593. 7243441; 41.								

* These high factorisations and high primes were determined by aid of the *Binary Canon Extension* (compiled by Mr. H. J. Woodall and the author), which gives the Residues of 2^x up to $x=100$ after division by every prime and prime-power ≥ 10000 .

The following Table is a *complete** List of small prime divisors (p) of H_n , up to $p \gtrsim 1400$, shewing the least suffixes n_0, n'_0 of H_n to each prime:—

p	17	41	97	113	137	193	313	353	401	409	449	521	569	577	593
n_0	1	2	17	3	24	13	46	4	67	70	103	24	108	63	16
n'_0	7	18	31	25	44	83	110	84	133	134	121	236	176	81	132

p	761	769	809	857	929	977	1009	1129	1153	1201	1217	1297	1361
n_0	54	41	134	60	181	155	7	122	84	86	5	35	49
n'_0	326	293	270	368	213	333	497	442	204	214	147	613	631

The following scheme shews the conditions† which—when n is possible under condition (108)—suffice to show whether n is *odd* or *even*; [(a, b), (c, d) being the 2^{10} “parts” of p , as in (104)]: hence this usually suffices when $p = 2^k \cdot q + 1$ with q prime up to $(2/p)_{16} = +1$, if $(2/p)_q \neq 1$.

<i>Residuacity</i>	d	$p-1$	b	ξ	n	<i>Residuacity</i>	d	$p-1$	b	ξ	n
$(2/p)_4 = -1$	2ω	8ε	4ω	ε	ω	$(2/p)_{16} = -1$	8ω	16ε	16ω	ε	ω
„ $= -1$	2ε	8ω	4ω	ε	ε	„ $= -1$	8ω	16ω	16ω	ε	.
„ $= +1$	2ω	8ω	4ε	ω, ε	.	„ $= -1$	8ε	16ε	16ω	ε	ε
„ $= +1$	2ε	8ε	4ε	ω, ε	<i>infra</i>	„ $= -1$	8ε	16ω	16ω	ε	.
$(2/p)_8 = -1$	4ω	8ε	8ω	ε	ω	„ $= +1$	8ω	16ω	16ε	ω	$\omega \& \varepsilon$
„ $= -1$	4ε	8ε	8ω	ε	ε	„ $= +1$	8ω	16ε	16ε	ω, ε	.
„ $= +1$	4ω	8ε	8ε	ω, ε	.	„ $= +1$	8ε	16ω	16ε	ω	.
„ $= +1$	4ε	8ε	ε	ω, ε	<i>infra</i>	„ $= +1$	8ε	16ε	16ε	ω, ε	?

36a. Assoc. Half-Quartans. Some of the numbers of type similar to (100b), (103),

$H_n = \frac{1}{2}(x'^4 + y'^4) = \frac{1}{2}\{(y^n - 1)^4 + (y^n + 1)^4\} = y^{4n} + 6y^{2n} + 1 \dots (111),$
are of interest from the simplicity of the forms (111) for

* It will be noticed that $n_0 + n'_0 = \xi$ is *even* throughout this Table: the *lowest* prime divisor of H_n with ξ *odd* is $p = 4721$, for which $n_0 = 57$, $n'_0 = 238$, $\xi = 295$.

† The conditions depending on $(2/p)_4$ have been *proved* by Mr. Th. Gosset: those under $(2/p)_8$ depend on induction from numerous examples; those under $(2/p)_{16}$ are somewhat conjectural (as they depend on only a few examples). The author's acknowledgments are due to Mr. Gosset for help in preparing Art. 36, and in drawing up this scheme.

certain values of y ; (e.g. $y = 6, 10, \&c.$). The 2^{ic} forms are

$$H_n = (y^{2n} + 1)^2 + (2y_n)^2 = (y^{2n} - 1)^2 + 2 \cdot 2y^n)^2 = (y^{2n} + 3)^2 - 2 \cdot 2^2 \dots (112).$$

The factorisation of the first few of these numbers with $y = 6, 10$ are given below.

n	$y=6$			$y=10$		
	x'	y'	H_n	x'	y'	H_n
1	5,	7	17.89;	9,	11	10601;
2	35,	37	1687393;	99,	101	100060001;
3	215,	217	641.3396353;	999,	1001	17.5569.10562737;

The Rules given for the divisors of H_n in Art. 36 apply, with slight changes here also. Thus, with the preliminaries laid down in Results (104) to (106), the Results (107) to (108) of that Article are applicable here also, after changing n, n', n_0, n'_0 thereof into $2n', 2n_0, 2n'_0$; (but the rest of the detail from Result (108) onwards does not apply).

37. $2^{ic}, 4^{ic}, \& 8^{ic}$ roots (combined). Let y_2, y_4, y_8 be roots of the $2^{ic}, 4^{ic}$, and 8^{ic} congruences

$$y_2^2 + 1 \equiv 0, y_4^4 + 1 \equiv 0, y_8^8 + 1 \equiv 0, \pmod{p} \dots\dots\dots (113).$$

Then will—

$$y_2^4 \equiv +1, y_4^4 \equiv -1; \quad y_2^8 \equiv +1, y_4^8 \equiv +1, y_8^8 \equiv -1 \pmod{p}.$$

And then will $N_4 = y_2^4 + y_4^4 \equiv 0, \pmod{p} \dots\dots\dots (113a),$

and $N_8 = y_2^8 + y_8^8 \equiv 0, \quad \& \quad N_8 = y_4^8 + y_8^8 \equiv 0, \pmod{p} \dots\dots\dots (113b).$

More generally, $N_4 = Y_2^4 + Y_4^4 \equiv 0, \pmod{p} \dots\dots\dots (114a),$

and $N_8 = Y_2^8 + Y_8^8 \equiv 0, \quad \& \quad N_8 = Y_4^8 + Y_8^8 \equiv 0, \pmod{p} \dots\dots\dots (114b),$

where $Y_2 = m_2 p \pm y_2, Y_4 = m_4 p \pm y_4, Y_8 = m_8 p \pm y_8 \dots\dots\dots (114c).$

Thus p is a divisor of the 4-tans and 8-vans (N_4, N_8) above written. Here p may be either prime or composite.

Note that if the pair of $y_2, y_4, y_8, Y_2, Y_4, Y_8$ entering into N_4 or N_8 contain a common factor, this should be rejected in forming N_4 or N_8 .

37a. Use of Congruence-Tables. The Tables of solutions (y_2, y_4, y_8) of the above congruences (113), described in Art. 8, 15, enable any number of 4-tans and 8-vans ($N_4, 1N_8$), which shall have a given divisor $p (= 8\pi + 1$ or $16\pi + 1)$, to be written down at once. And, when the quotient Q of

$N \div p$ is $\gt 9$ million, the complete factorisation of all such numbers (N_4 , N_8) can be effected by the large Factor-Tables.

[This Method is not very convenient for finding factors of given numbers N_4 , N_8 , and is not suited for detecting High Primes (as it is not exhaustive).

Ex. The following examples of 4-tans and 8-vans (N_4 , N_8) have been prepared and factorised by aid of the Congruence Tables mentioned.

μ denotes the factor, common to y_2 , y_4 , y_8 (as given by the Tables), rejected in forming N_4 and N_8 .

y_2	y_4	y_8	p	μ	N_4 or N_8
86,	277,	.	569	.	} $277^4 + 86^4 = 569.7673.1361$;
277,	86,	.	7673	.	
.	19,	10	17	.	} $19^8 + 10^8 = 17.3361.298993$;
.	30,	57	3361	3	
.	260,	300	4513	20	} $13^8 + 15^8 = 2.4513.374321$;
.	19,	23	17	.	
.	323,	391	641	17	} $19^8 + 23^8 = 2.17.641.4372513$;

38. Semi-quartic forms. The question arises whether Quartans, &c., can be expressed in any of the semi-quartic forms

$$N \text{ or } \frac{1}{2}N = t_1^4 - 2u_1^2 = t_2^2 - 2u_2^4 = 2u_3^4 - t_3^2 = 2u_4^2 - t_4^4 \dots (115).$$

These will be denoted Forms i., ii., iii., iv. for brevity:—

On referring to the 2^{ic} forms (Art. 4. 4b), it is seen that this can sometimes be done *algebraically*:—

If $N = x^4 + y^4$, then—

iv. By (3d), $N = 2u_4^2 - t_4^4$, in two ways, always(115a).

i. By (3c), $N = t_1^4 - 2u_1^2$, if $x^2 + y^2 = t_1^2$ (115b).

iii. By (4d), $\frac{1}{2}N = 2u_2^4 - t_2^2$, if $x^2 + y^2 = 2u_2^2$ (115c).

i. By (4c), $\frac{1}{2}N = t_1^4 - 2u_1^2$, if $(x^2 \mp xy + y^2) = t_1$ (115d).

And, if $N = x^8 + y^8$, then

ii. By (3c), $N = t_2^2 - 2u_2^4$, always(116a).

iv. By (3d), $N = 2u_4^2 - t_4^4$, in two ways, always(116b).

iv. By (4d), $\frac{1}{2}N = 2u_2^4 - t_2^2$, always(116c).

Except, as above, the semi-quartic partitions are by no means easy to form, and there appear to be no known rules as to their possibility.

Examples. Here follow some examples of each of the cases i, iii, i, under the conditions named

Condition.	x, y	N or $\frac{1}{2}N$.	x, y	N or $\frac{1}{2}N$.
i. $x^2+y^2=t_1^2$	3, 4	$327=5^4-2.12^2$	5, 12	$21361=13^4-2.60^2$
iii. $x^2+y^2=2u_2^2$	7, 1	$1201=2.5^4-7^2$	17, 7	$42961=2.13^4-119^2$
i. $x^2 \mp xy + y^2 = t_1^2$	5, 3	$353=7^4-2.32^2$	15, 7	$26573=13^4-2.32^2$

The following Table shows all the Half-Quartan primes > 5000 expressed in form iii., from which it would seem *probable* that they are all so expressible (at any rate arithmetically).

$x, y = 1, 1; 3, 1; 5, 1; 5, 3; 7, 1; 9, 5; 9, 7;$
$\frac{1}{2}N = 1; 41; 313; 353; 1201; 3593; 4481;$
$u, t = 1, 1; 3, 11; 7, 67; 9, 113; 5, 7; 19, 507; 45, 2863;$

39. Quartic Problems. From the Results of Art. 38, a variety of interesting quartic problems may be solved.

39a. Problem i. To solve

$$x^4 + y^4 + z^4 = 2u^2 \dots \dots \dots (117).$$

This is the same thing as solving [see (3d) & (115a)]

$$N = x^4 + y^4 = 2u^2 - z^4 \\ = 2(x^2 + xy + y^2)^2 - (x + y)^4 \dots \dots \dots (117a).$$

Take u any quantity of form

$$u = A^2 + 3B^2 = \{(B-A)^2 + (B-A)(B+A) + (B+A)^2\} \dots \dots (118).$$

Then $x = B - A, \quad y = B + A, \quad z = x + y = 2B \dots \dots \dots (118a).$

Of course x, y, z are *interchangeable* in the above: and evidently either (A, B) or any two of (x, y, z) may be assumed at pleasure.

39b. Problem ii. To solve*

$$N = x^4 + y^4 + z^4 = x'^4 + y'^4 + z'^4 = x''^4 + y''^4 + z''^4 = \&c. \dots \dots (119).$$

Write $N = 2u^2 = 2u'^2 = 2u''^2 = \&c. \dots (119a),$

and take $u = u' = u'' = \&c.$ any *composite* number expressible in several ways in the forms

$$u = A^2 + 3B^2, \quad u' = A'^2 + 3B'^2, \quad u'' = A''^2 + 3B''^2 = \&c. \dots \dots (120).$$

Then, as in Art. 39a,

$$x = B - A, \quad y = B + A, \quad z = x + y = 2B \dots \dots \dots (120a),$$

* This solution was given by the author in the *Educ. Times* of May, 1903 p. 228, (Reply to Q. 16337).

with similar results obtained by simply accentuating all the symbols. And, in general, N will be expressible in the 4-ic form (119) in the same number of ways as there are expressions of u in the 2^{ic} form (120).

Examples. The following Table shows the results obtained by using the small composites

$$N=91=7.13; \quad 133=7.19; \quad 247=13.19; \quad 2 \text{ forms each,}$$

$$N=1729=7.13.19; \quad 4 \text{ forms.}$$

N	A, B	x, y, z	N	A, B	x, y, z
91=7.13	{	8, 3; 5, 11, 6 4, 5; 1, 9, 10	1729=	{	41, 4; 37, 45, 8 31, 16; 15, 47, 32
133=7.19	{	11, 2; 9, 13, 4 5, 6; 1, 11, 12	7.13.19	{	23, 20; 3, 43, 40 1, 24; 23, 25, 48
247=13.19	{	10, 7; 3, 17, 14 2, 9; 7, 11, 18			

39c. Problem iii. To solve* the simultaneous equations

$$N_1 = \xi^8 + \eta^8 + \zeta^8 = x'^4 + y'^4 + z'^4, \quad N_2 = \xi^4 + \eta^4 + \zeta^4 = x'^2 + y'^2 + z'^2 \dots (121).$$

$$\text{Take } \xi, \eta, \zeta \text{ such that } \xi^2 + \eta^2 = \zeta^2, [\xi \text{ odd, } \eta \text{ even}] \dots (122).$$

And, in the solution of Prob. i, write $x = \xi^2, y = \eta^2, z = \zeta^2$.

$$\text{Then} \quad N_1 = \xi^8 + \eta^8 + \zeta^8 = 2u^2 \dots (123),$$

$$\text{where} \quad u = \xi^4 + \xi^2\eta^2 + \zeta^4 = (\xi^2 - \xi\eta + \zeta^2)(\xi^2 + \xi\eta + \zeta^2) \dots (123a),$$

so that here u is always *composite*, and can therefore be expressed in two ways in the forms $u = A^2 + 3B^2, u' = A'^2 + 3B'^2$, as in Prob. ii.

In fact

$$u = (\xi^2 + \frac{1}{2}\eta^2)^2 + 3(\frac{1}{2}\eta^2)^2 = \xi^4 + \xi^2\eta^2 + \eta^4, [\text{as above}] \dots (124a),$$

$$u' = (\xi^2 - \eta^2)^2 + 3(\xi\eta)^2 = x'^2 + x'y' + y'^2, [\text{as in Prob. ii.}] \dots (124b),$$

$$\text{where} \quad x' = \xi^2 - \xi\eta - \eta^2, \quad y' = 2\xi\eta, \quad z' = x' + y' = \xi^2 + \xi\eta - \eta^2 \dots (124c).$$

Hence, as in Prob. ii.

$$N_1 = \xi^8 + \eta^8 + \zeta^8 = x'^4 + y'^4 + z'^4 \dots (125).$$

$$\text{Next, take} \quad N_2 = \xi^4 + \eta^4 + \zeta^4, [\text{with same } \xi, \eta, \zeta \text{ as above}] \dots (126),$$

$$= \xi^4 + \eta^4 + (\xi^2 + \eta^2)^2 = 2(\xi^4 + \xi^2\eta^2 + \eta^4) = 2u.$$

$$\text{Hence} \quad N_2 = 2u = 2u' = 2(x'^2 + x'y' + y'^2)$$

$$= x'^2 + y'^2 + (x' + y')^2 = x'^2 + y'^2 + z'^2.$$

$$\text{Thus} \quad N_2 = \xi^4 + \eta^4 + \zeta^4 = x'^2 + y'^2 + z'^2, \text{ as required } \dots (127).$$

This seems a remarkable result that the same values of $\xi, \eta, \zeta, x', y', z'$, should satisfy both N_1, N_2 at once.

* This solution was given by the author in the *Educator*, Times of April, 1908, p. 188, (Reply to Q 16368): the problem was proposed by Prof. Escott.

Ex. Since $3^2 + 4^2 = 5^2$, and $5^2 + 12^2 = 13^2$,

$$N_1 = 3^2 + 4^2 + 5^2 = 5^2 + 19^2 + 24^2, \quad N_2 = 3^4 + 4^4 + 5^4 = 5^2 + 19^2 + 29^2,$$

$$N_1 = 5^2 + 12^2 + 13^2 = 59^2 + 120^2 + 179^2, \quad N_2 = 5^4 + 12^4 + 13^4 = 59^2 + 120^2 + 179^2.$$

39d. Problem iii. To solve

$$(\xi^4 + \eta^4 + \zeta^4)^2 = 2(\xi^8 + \eta^8 + \zeta^8) \dots\dots\dots (128a),$$

and

$$(x^2 + y^2 + z^2)^2 = 2(x^4 + y^4 + z^4) \dots\dots\dots (128b).$$

In Problem iii., $N_1 = 2u^2$ and $N_2 = 2u$, whence $N_2^2 = 2N_1$: this solves* the present Problem.

39e. Problem iv. To solve

$$N = x^4 + y^4 + z^4 = 2u^{2n} \dots\dots\dots (129).$$

Take u any quantity of form $u = A^2 + 3B^2$. Then u^{2n} may be expressed in the form

$$u^{2n} = A^2 + 3B^2 \dots\dots\dots (130),$$

by known rules: whenever, as in Art. 39a,

$$x = B - A, \quad y = B + A, \quad z = x + y = 2B \dots\dots\dots (131).$$

Cor. If u be taken *composite*, then $2u^{2n}$ may be expressed in the 4^{ic} form (129) in as many ways as u can be expressed in the 2^{ic} form (130).

39f. Problem iv. To solve

$$N = x^4 + y^4 + z^4 = 2u^4 \dots\dots\dots (132).$$

This is a particular case of Problem iv. above. Taking

$$u = A^2 + 3B^2, \quad u^2 = A^2 + 3A^2 \dots\dots\dots (133),$$

where

$$A = A^2 + 3B^2, \quad B = 2AB \dots\dots\dots (133a),$$

whence x, y, z are given by (131).

39g. Problem v. To solve

$$N = x^4 + y^4 + 2w^4 = x'^4 + y'^4 + 2w'^4 \dots\dots\dots (134).$$

Here it suffices to solve Problem iv. *twice*, so that

$$x^4 + y^4 + z^4 = 2u^4, \quad x'^4 + y'^4 + z'^4 = 2u'^4 \dots\dots\dots (135),$$

taking

$$u = A^2 + 3B^2, \quad u' = A'^2 + 3B'^2 \dots\dots\dots (136),$$

with the condition

$$AB = A'B', \quad \text{whence } z = z' \dots\dots\dots (136a),$$

whence

$$x^4 + y^4 + 2u'^4 = x'^4 + y'^4 + 2u^4 \dots\dots\dots (137),$$

By taking Y, B *composite*, it is evident that numbers N can be found expressible in the 4^{ic} form (134) in as many ways as AB admits of resolution into two co-factors.

* This solution was given in the *Educ. Times* for September, 1908, p. 411 (Reply to Q. 16422).

Examples. Each line below is one example of Prob. v.—(or two of Prob. iv.)—obtained by simply interchanging A, B —($A' = B, B' = A$);

[Note that for Prob. v., $u = w', u' = w$].

$A, B;$	$A, B;$	x, y, z, u	$A', B';$	$A', B';$	x', y', z', u'
2, 1;	1, 4;	3, 5, 8, 7	1, 2;	11, 4;	7, 15, 8, 13;
4, 1;	13, 8;	5, 21, 16, 19	1, 4;	47, 8;	39, 55, 16, 49;

39h. Problem vi. To solve

$$N = \text{Sum of } n \text{ 4th powers} = \text{Sum of } n \text{ 4th powers} \dots (138).$$

This may be easily solved by repeating and combining, as may be necessary, several solutions of

$$X^4 + Y^4 = X'^4 + Y'^4 \text{ and } x^4 + y^4 + z^4 = x'^4 + y'^4 + z'^4 \dots (138a).$$

But a more interesting solution is also obtainable which cannot be broken up into smaller equal sums (*i.e.* of less than 4 quantities).

Thus, taking $n = 4$, to solve—

$$N = X^4 + Y^4 + Z^4 + W^4 = X'^4 + Y'^4 + Z'^4 + W'^4 \dots (139).$$

Take any solution of $X^4 + Y^4 = X'^4 + Y'^4$,
and form a solution of $x^4 + y^4 + z^4 = x'^4 + y'^4 + z'^4$,
in such a way that one of X', Y' = one of x, y, z ; say $Y' = y$,
then will

$$X^4 + Y^4 + x^4 + z^4 = X'^4 + x'^4 + y'^4 + z'^4 \dots (139a),$$

a solution which cannot be broken up into two sets of two equal sums of two 4th powers each.

Ex. Taking Euler's example in smallest numbers (Art 27d)

$$133^4 + 134^4 = 158^4 + 59^4; \text{ take } Y' = 59;$$

Now take $y = 59 = B + A$, and take $A = 44, B = 15$.

$$\text{Then } A^4 + 3B^4 = 44^4 + 3 \cdot 15^4 = 2611 = 32^4 + 3 \cdot 23^4 = A'^4 + 3B'^4.$$

$$\text{Hence, by Prob. v. } 29^4 + 59^4 + 30^4 = 2 \cdot 2611^2 = 9^4 + 55^4 + 46^4.$$

Adding (and omitting 59^4);

$$133^4 + 134^4 + 29^4 + 30^4 = 158^4 + 9^4 + 55^4 + 46^4,$$

which cannot be broken up into equal sums of two 4th powers each.

Similarly two solutions of $x^4 + y^4 + z^4 = x'^4 + y'^4 + z'^4$, which contain one of (x, y, z, x', y', z') in common, can be combined so as to yield two equal sums of five 4th powers (with no two equal).

Ex. Combining the 2nd and 3rd examples of Prob. ii., which have the common element 11^4 , yields

$$4^4 + 7^4 + 9^4 + 13^4 + 18^4 = 1^4 + 3^4 + 12^4 + 14^4 + 17^4.$$

(To be continued.)

A THEOREM OF CAYLEY'S REGARDING STURM'S FUNCTIONS.

By Thomas Muir, LL.D.

(1) IN *Liouville's Journal*, vol. XIII., pp. 269–274, Cayley enunciated a series of identities which he himself described as remarkable, but which he warned his readers he had not succeeded in proving. Denoting the functions

$$\left. \begin{aligned} &ax^n + bx^{n-1} + cx^{n-2} + \dots \\ &a'x^n + b'x^{n-1} + c'x^{n-2} + \dots \end{aligned} \right\} \text{by } \begin{cases} V \\ V' \end{cases};$$

the derived functions

$$\left| \begin{array}{cc} V & V' \\ a & a' \end{array} \right|, \quad \left| \begin{array}{cccc} xV & V & xV' & V' \\ a & . & a' & . \\ b & a & b' & a' \\ c & b & c' & b' \end{array} \right|, \quad \left| \begin{array}{cccccc} x^2V & xV & V & x^2V' & xV' & V' \\ a & . & . & a' & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \end{array} \right|,$$

&c., by $-F_1, F_2, -F_3, \dots$; the determinants

$$\left| \begin{array}{cc} a & a' \\ b & b' \end{array} \right|, \quad \left| \begin{array}{cccc} a & . & a' & . \\ b & a & b' & a' \\ c & b & c' & b' \\ d & c & d' & c' \end{array} \right|, \quad \left| \begin{array}{cccccc} a & . & . & a' & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \\ f & e & d & f' & e' & d' \end{array} \right|,$$

&c., by $P_1, P_2, P_3, \dots$; and by $P'_1, P'_2, P'_3, \dots$, the determinants obtained from these latter by substituting

$$c \ c, \quad e \ d \ e' \ d', \quad g \ f \ e \ g' \ f' \ e', \dots$$

for the last rows, Cayley asserted that

$$P_1^2 F_3 + (xP_1 P_2 + P_1 P'_2 + P'_1 P_2) F_2 + P_2^2 F_1 = 0,$$

$$P_2^2 F_4 + (xP_2 P_3 + P_2 P'_3 + P'_2 P_3) F_3 + P_3^2 F_2 = 0,$$

.....

A careful examination has established the fact that the theorem as thus stated is seriously inaccurate, and that, to

rectify it, it is necessary (1) to denote the first series of determinants

not by $-F_1, F_2, -F_3, F_4, -F_5, F_6, -F_7, F_8, -F_9, \dots$,

but by $-F_1, F_2, F_3, -F_4, -F_5, F_6, F_7, -F_8, -F_9, \dots$;

(2) to make the middle term of the trinomial coefficient *negative*; and (3) to make the signs preceding the said trinomial coefficients $+$ and $-$ alternately. Proof of this could be given in Cayley's own notation, and—what of course is most important—would involve the missing demonstration: indeed it was while searching for the said demonstration that the inaccuracies were detected. I prefer, however, to put the whole matter in a new form, for the purpose not only of securing simplicity but of being enabled to state some additional results which a consideration of the process of demonstration has brought to light.

(2) Let us denote the initial functions by F_n, G_n , the suffix thus indicating the degree in x ; and let us use the same capital letters, with different suffixes, to stand for certain derived functions, namely

$$F_{n-1} \equiv \begin{vmatrix} a & F_n \\ a' & G_n \end{vmatrix} = \begin{vmatrix} a & bx^{n-1} + cx^{n-2} + \dots \\ a' & b'x^{n-1} + c'x^{n-2} + \dots \end{vmatrix} = 2_0x^{n-1} + 2_1x^{n-2} + \dots,$$

$$G_{n-1} \equiv \begin{vmatrix} a & b & xF_n \\ a' & b' & xG_n \\ . & a & F_n \end{vmatrix} = \begin{vmatrix} a & b & cx^{n-1} + \dots \\ a' & b' & c'x^{n-1} + \dots \\ . & a & bx^{n-1} + \dots \end{vmatrix} = 3_0x^{n-1} + 3_1x^{n-2} + \dots,$$

$$F_{n-2} \equiv \begin{vmatrix} a & b & c & xF_n \\ a' & b' & c' & xG_n \\ . & a & b & F_n \\ . & a' & b' & G_n \end{vmatrix} = \begin{vmatrix} a & b & c & dx^{n-2} + \dots \\ a' & b' & c' & d'x^{n-2} + \dots \\ . & a & b & cx^{n-2} + \dots \\ . & a' & b' & c'x^{n-2} + \dots \end{vmatrix} = 4_0x^{n-2} + 4_1x^{n-3} + \dots,$$

$$G_{n-2} \equiv \begin{vmatrix} a & b & c & d & x^2F_n \\ a' & b' & c' & d' & x^2G_n \\ . & a & b & c & xF_n \\ . & a' & b' & c' & xG_n \\ . & . & a & b & F_n \end{vmatrix} = \begin{vmatrix} a & b & c & d & ex^{n-2} + \dots \\ a' & b' & c' & d' & e'x^{n-2} + \dots \\ . & a & b & c & dx^{n-2} + \dots \\ . & a' & b' & c' & d'x^{n-2} + \dots \\ . & . & a & b & cx^{n-2} + \dots \end{vmatrix} = 5_0x^{n-2} + 5_1x^{n-3} + \dots,$$

.....

On looking into these it will be seen that $F_{n-1}, F_{n-2}, \dots$ are Cayley's functions $F_1, F_2, \dots$; that $G_{n-1}, G_{n-2}, \dots$ are functions defined in a similar manner and furnishing the odd-ordered determinants wanting from Cayley's series; that the second determinants are simplifications of Cayley's, which are easily remembered and which have the further advantage of suggesting a very useful polynomial form with suffixed integers for coefficients.

The derived functions are got by performing on F_n and G_n , the division-process for finding their highest common factor, namely

$$\begin{aligned} & F_n) a G_n (a' \\ & \quad \underline{a' F_n} \\ & \quad F_{n-1}) 2_0 F_n (ax \\ & \quad \quad \underline{ax F_{n-1}} \\ & \quad \quad G_{n-1}) 3_0 F_{n-1} (2_0 \\ & \quad \quad \quad \underline{2_0 G_{n-1}} \\ & \quad \quad \quad a F_{n-2}) 4_0 G_{n-1} (3_0 x \\ & \quad \quad \quad \quad \underline{3_0 x F_{n-2}} \\ & \quad \quad \quad \quad 2_0 G_{n-2}) 5_0 F_{n-2} (4_0 \\ & \quad \quad \quad \quad \quad \underline{4_0 G_{n-2}} \\ & \quad \quad \quad \quad \quad 3_0 F_{n-3}) 6_0 G_{n-2} (5_0 x \\ & \quad \quad \quad \quad \quad \quad \dots\dots\dots, \end{aligned}$$

where, to show the full sphere of the 'reign of law,' we might have used 1_0 for a , as the definition of our notation of course entitles us to do. The process is seen to imply the existence of the equations

$$\begin{aligned} a G_n - a' F_n &= F_{n-1}, \\ 2_0 F_n - ax F_{n-1} &= G_{n-1}, \\ 3_0 F_{n-1} - 2_0 G_{n-1} &= a F_{n-2}, \\ 4_0 G_{n-1} - 3_0 x F_{n-2} &= 2_0 G_{n-2}, \\ 5_0 G_{n-2} - 4_0 G_{n-2} &= 3_0 F_{n-3}, \\ &\dots\dots\dots \end{aligned}$$

and will be found to imply also the use of the identities

$$\begin{aligned} 2_1 3_0 - 2_0 3_1 &= 1_0 4_0, \\ 3_1 4_0 - 3_0 4_1 &= 2_0 5_0, \\ 4_1 5_0 - 4_0 5_1 &= 3_0 6_0, \\ &\dots\dots\dots \end{aligned}$$

a specimen one of which, the second, we may establish by noting that Laplace's expansion-theorem gives

$$\begin{aligned}
 -4_0 3_1 + 4_1 3_0 &= \begin{vmatrix} a & a' & . & . & . & . & . \\ b & b' & a & a' & . & . & . \\ c & c' & b & b' & . & . & . \\ d & d' & c & c' & c & c' & b \\ e & e' & d & d' & d & d' & c \\ . & . & . & . & b & b' & a \\ . & . & . & . & a & a' & . \end{vmatrix} = \begin{vmatrix} a & a' & . & . & . & . & . \\ b & b' & a & a' & a & a' & . \\ c & c' & b & b' & b & b' & a \\ d & d' & c & c' & c & c' & b \\ e & e' & d & d' & d & d' & c \\ . & . & . & . & b & b' & a \\ . & . & . & . & a & a' & . \end{vmatrix} \\
 &= \begin{vmatrix} a & a' & . & . & . & . & . \\ b & b' & a & a' & . & . & . \\ c & c' & b & b' & . & . & a \\ d & d' & c & c' & . & . & b \\ e & e' & d & d' & . & . & c \\ . & . & . & . & b & b' & a \\ . & . & . & . & a & a' & . \end{vmatrix} = 2_0 5_0.
 \end{aligned}$$

3. Taking the third, fourth, and fifth of our equations of derivation, and eliminating G_{n-1} , G_{n-2} , we have

$$\begin{vmatrix} -1_0 F_{n-3} + 3_0 F_{n-1} & -2_0 & . \\ -3_0 x F_{n-2} & 4_0 & -2_0 \\ -3_0 F_{n-3} + 5_0 F_{n-2} & . & -4_0 \end{vmatrix} = 0$$

$$\text{or } 0 = 2_0^2 3_0 F_{n-3} - (2_0 3_0 4_0 x + 1_0 4_0^2 + 5_0 2_0^2) F_{n-2} + 3_0 4_0^2 F_{n-1}.$$

But from what we have already seen

$$\begin{aligned}
 1_0 4_0^2 + 5_0 2_0^2 &= 4_0 (2_1 3_0 - 2_0 3_1) + 2_0 (3_1 4_0 - 3_0 4_1) \\
 &= 3_0 (4_0 2_1 - 4_1 2_0):
 \end{aligned}$$

Consequently the equation connecting F_{n-3} , F_{n-2} , F_{n-1} takes the form

$$2_0^2 F_{n-3} - (2_0 4_0 x - 2_0 4_1 + 2_1 4_0) F_{n-2} + 4_0^2 F_{n-1} = 0:$$

and this is Cayley's first identity. The second

$$4_0^2 F_{n-4} - (4_0 6_0 x - 4_0 6_1 + 4_1 6_0) F_{n-3} + 6_0^2 F_{n-2} = 0$$

is got in exactly the same way by using the 5th, 6th, and 7th of the equations of derivation: and so on for the others. They

show, as Cayley desired, that F_{n-1} and F_{n-2} are of opposite signs when x has any value that makes F_{n-2} vanish; that F_{n-2} and F_{n-4} are of opposite signs when x has any value that makes F_{n-3} vanish: and so on.

4. The identities can, however, be proved in a totally different way, by ignoring altogether the mode of derivation of the functions F_{n-1} , F_{n-2} , ..., and simply substituting for the said functions the polynomial expressions given as their equivalents in § 2. For example, we then have

$$\begin{aligned} & 4_0^2 F_{n-1} - (2_0 4_0 x - 2_0 4_1 + 2_1 4_0) F_{n-2} + 2_0^2 F_{n-3} \\ &= 4_0^2 \{ 2_0 x^{n-1} + 2_1 x^{n-2} + 2_2 x^{n-3} + \dots \} \\ & \quad - (2_0 4_0 x - 2_0 4_1 + 2_1 4_0) (4_0 x^{n-2} + 4_1 x^{n-3} + 4_2 x^{n-4} + \dots) \\ & \quad + 2_0^2 \{ 6_0 x^{n-3} + 6_1 x^{n-4} + 6_2 x^{n-5} + \dots \} \\ &= x^{n-1} (4_0^2 2_0 - 2_0 4_0^2) \\ & \quad + x^{n-2} (4_0^2 2_1 - 2_0 4_0 4_1 + 2_0 4_1 4_0 - 2_1 4_0 4_0) \\ & \quad + x^{n-3} (4_0^2 2_2 - 2_0 4_0 4_2 + 2_0 4_1 4_1 - 2_1 4_0 4_1 + 2_0^2 6_0) \\ & \quad + x^{n-4} (4_0^2 2_3 - 2_0 4_0 4_3 + 2_0 4_1 4_2 - 2_1 4_0 4_2 + 2_0^2 6_1) \\ & \quad + \dots \end{aligned}$$

Here the coefficients of x^{n-1} and x^{n-2} manifestly vanish, and we have to show that the remaining coefficients vanish also, or generally that

$$4_0^2 2_r - 2_0 4_0 4_r + 2_0 4_1 4_{r-1} - 2_1 4_0 4_{r-1} + 2_0^2 6_{r-2} = 0.$$

Now this is merely the result of eliminating a determinant from two identities, namely, in the case of the coefficient of x^{n-3} the determinant $(abde)$ from

$$\left. \begin{aligned} 2_1 4_0 - 2_1 4_1 + 2_0 (abde) &= 0 \\ 4_2 4_0 - 4_1 4_1 + 4_0 (abde) &= 0 \end{aligned} \right\};$$

and these identities can be established by the use of Laplace's expansion-theorem exactly as in § 2. Thus

$$2_0 (abde) - 2_1 4_1 + 2_2 4_0 = - \begin{vmatrix} a & a' & . & . & . & . \\ b & b' & c & c' & b & b' \\ c & c' & d & d' & c & c' \\ d & d' & e & e' & d & d' \\ . & . & b & b' & a & a' \\ . & . & a & a' & . & . \end{vmatrix}$$

and all that remains is to perform the operations $\text{row}_1 + \text{row}_5$, $\text{col}_6 - \text{col}_2$, $\text{col}_5 - \text{col}_1$. Similarly

$$4_0(abde) - 4_14_1 + 4_24_0 = \begin{vmatrix} a & a' & . & . & . & . & . & . \\ b & b' & a & a' & . & . & . & . \\ c & c' & b & b' & . & . & . & . \\ d & d' & c & c' & c & c' & b & b' \\ e & e' & d & d' & d & d' & c & c' \\ f & f' & e & e' & e & e' & d & d' \\ . & . & . & . & b & b' & a & a' \\ . & . & . & . & a & a' & . & . \end{vmatrix}$$

with the operations $\text{row}_2 + \text{row}_8$, $\text{row}_3 + \text{row}_7$, $\text{col}_5 - \text{col}_3$, $\text{col}_6 - \text{col}_4$ to follow.

5. The first of these two auxiliary identities belongs to the series

$$2_24_0 - 2_14_1 + 2_0(abde) = 0,$$

$$4_26_0 - 4_16_1 + 4_0(abcdefg) = 0,$$

$$6_28_0 - 6_18_1 + 6_0(a...fhi) = 0,$$

.....,

which is included in the series of 'extensional' identities

$$\begin{aligned} & |b_1c_2| \cdot |b_1c_3d_4e_5| - |b_1c_3| \cdot |b_1c_2d_4e_5| \\ & + |b_1c_4| \cdot |b_1c_2d_3e_5| - |b_1c_5| \cdot |b_1c_2d_3e_4| = 0, \text{ \&c.;} \end{aligned}$$

and the second belongs to the series

$$4_24_0 - 4_14_1 + 4_0(abde) = 2_06_0,$$

$$6_26_0 - 6_16_1 + 6_0(abcd fg) = 4_08_0,$$

.....

which is included in the series of 'extensional' identities

$$\begin{aligned} & |a_1b_2c_3d_4| \cdot |b_3c_4e_5f_6| - |a_1b_2c_3e_4| \cdot |b_3c_4d_5f_6| + |a_1b_2c_3f_4| \cdot |b_3c_4d_5e_6| \\ & - |b_1c_2d_3e_4| \cdot |a_3b_4c_5f_6| + |b_1c_2d_3f_4| \cdot |a_3b_4c_5e_6| - |b_1c_2e_3f_4| \cdot |a_3b_4c_5d_6| \\ & = |b_3c_4| \cdot |a_1b_2c_3d_4e_5f_6|, \text{ \&c.} \end{aligned}$$

6. Returning now to § 2 and taking the 4th, 5th, and 6th

of the equations of derivation, we obtain on eliminating F_{n-2}, F_{n-3}

$$\begin{vmatrix} . & -3_0x & -2_0G_{n-2}+4_0G_{n-1} \\ -3_0 & 5_0 & -4_0G_{n-1} \\ -5_0x & . & 4_0G_{n-3}+6_0G_{n-2} \end{vmatrix} = 0,$$

and therefore

$$3_0^2 4_0 G_{n-3} - (3_0 4_0 5_0 x + 2_0 5_0^2 + 6_0 3_0^2) G_{n-2} + 4_0 5_0^2 G_{n-1} = 0,$$

$$i.e. \quad 3_0^2 G_{n-3} - (3_0 5_0 x - 3_0 5_1 + 3_1 5_0) G_{n-2} + 5_0^2 G_{n-1} = 0.$$

We thus see that there exist equations, connecting every three consecutive G 's, exactly similar to Cayley's connecting every three consecutive F 's, and showing, like Cayley's, that when for any particular value of x one of the G 's vanishes, the G preceding and the G following have for that value of x different signs. In fact instead of viewing this new series of identities as separate from but analogous to Cayley's there are certain advantages in considering the two series as forming only one series, namely

$$2_0^2 F_{n-3} - (2_0 4_0 x - 2_0 4_1 + 2_1 4_0) F_{n-2} + 4_0^2 F_{n-1} = 0,$$

$$3_0^2 G_{n-3} - (3_0 5_0 x - 3_0 5_1 + 3_1 5_0) G_{n-2} + 5_0^2 G_{n-1} = 0,$$

$$4_0^2 F_{n-4} - (4_0 6_0 x - 4_0 6_1 + 4_1 6_0) F_{n-3} + 6_0^2 F_{n-2} = 0,$$

$$5_0^2 G_{n-4} - (5_0 7_0 x - 5_0 7_1 + 5_1 7_0) G_{n-3} + 7_0^2 G_{n-2} = 0,$$

.....

the even-ordered determinants being associated with the F 's and the odd-ordered with the G 's.

Capetown, S.A.,
18th May, 1908.

ON THE HARMONIC CONIC OF TWO GIVEN CONICS.

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LET the equation of a conic circumscribing the triangle of reference be

$$S \equiv lyz + mzx + nxy = 0.$$

This conic may be regarded as the isogonal transformation of the line

$$L \equiv lx + my + nz = 0;$$

conversely, any line which does not pass through a vertex of the triangle of reference will transform isogonally into a conic circumscribing the triangle of reference. In particular the tangent to S at any point P will transform into a conic touching L at the point which is the isogonal conjugate of the point P .

Let now two conics be taken, viz.

$$S_1 \equiv 2l_1yz + 2m_1zx + 2n_1xy = 0,$$

$$S_2 \equiv 2l_2yz + 2m_2zx + 2n_2xy = 0.$$

These are respectively the isogonal transformations of the lines

$$L_1 \equiv l_1x + m_1y + n_1z = 0,$$

$$L_2 \equiv l_2x + m_2y + n_2z = 0.$$

Let a common tangent to S_1 and S_2 touch them in P' and P'' respectively and let its equation be

$$\lambda x + \mu y + \nu z = 0.$$

This transforms isogonally into the conic

$$\lambda yz + \mu zx + \nu xy = 0,$$

which touches L_1 and L_2 in the points which are the isogonal conjugates of P' and P'' respectively. Hence, since this conic has double contact with L_1 and L_2 , its equation is of the form

$$(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) - (px + qy + rz)^2 = 0.$$

Comparing this with the form

$$\lambda yz + \mu zx + \nu xy = 0,$$

we have $p^2 = l_1l_2$, $q^2 = m_1m_2$, $r^2 = n_1n_2$,

and this shows that the four conics which are the isogonal transformations of the common tangents of S_1 and S_2 touch L_1 and L_2 along the four lines

$$\sqrt{l_1l_2}x \pm \sqrt{m_1m_2}y \pm \sqrt{n_1n_2}z = 0.$$

These four lines determine on L_1 and L_2 eight points which are the isogonal conjugates of the eight points of contact of the four common tangents of S_1 and S_2 . The four points lying on L_1 and the four lying on L_2 may be regarded as the intersection of L_1 and L_2 with a quartic curve.

The equation of this curve will be of the form

$$L_1L_2(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + c_1c_2c_3c_4 = 0,$$

$$\begin{aligned}\text{where } c_1 &\equiv \sqrt{l_1 l_2} x + \sqrt{m_1 m_2} y + \sqrt{n_1 n_2} z = 0, \\ c_2 &\equiv \sqrt{l_1 l_2} x - \sqrt{m_1 m_2} y - \sqrt{n_1 n_2} z = 0, \\ c_3 &\equiv -\sqrt{l_1 l_2} x + \sqrt{m_1 m_2} y - \sqrt{n_1 n_2} z = 0, \\ c_4 &\equiv -\sqrt{l_1 l_2} x - \sqrt{m_1 m_2} y + \sqrt{n_1 n_2} z = 0.\end{aligned}$$

If we now let $a = l_1 l_2$, $b = m_1 m_2$, $c = n_1 n_2$,

$$2f = -(m_1 n_2 + m_2 n_1), \quad 2g = -(n_1 l_2 + n_2 l_1), \quad 2h = -(l_1 m_2 + l_2 m_1),$$

The quartic reduces to

$$\begin{aligned}& (m_1 n_2 - m_2 n_1)^2 y^2 z^2 + (n_1 l_2 - n_2 l_1)^2 z^2 x^2 + (l_1 m_2 - l_2 m_1)^2 x^2 y^2 \\ & + 2 (n_1 l_2 + n_2 l_1) (l_1 m_2 + l_2 m_1) x^2 y z + 2 (l_1 m_2 + l_2 m_1) (m_1 n_2 + m_2 n_1) x y^2 z \\ & + 2 (m_1 n_2 + m_2 n_1) (n_1 l_2 + n_2 l_1) x y z^2 = 0.\end{aligned}$$

The isogonal transformation of this—on which lie the eight points of contact of the common tangents of S_1 and S_2 —is the conic

$$\begin{aligned}& (m_1 n_2 - m_2 n_1)^2 x^2 + (n_1 l_2 - n_2 l_1)^2 y^2 + (l_1 m_2 - l_2 m_1)^2 z^2 \\ & + 2 (n_1 l_2 + n_2 l_1) (l_1 m_2 + l_2 m_1) y z + 2 (l_1 m_2 + l_2 m_1) (m_1 n_2 + m_2 n_1) z x \\ & + 2 (m_1 n_2 + m_2 n_1) (n_1 l_2 + n_2 l_1) x y = 0,\end{aligned}$$

which is easily shown to be the F or harmonic conic of S_1 and S_2 .

The form of the above quartic also shows that on the conic F lie eight points which are the isogonal conjugates of the points in which the four lines

$$\sqrt{l_1 l_2} x \pm \sqrt{m_1 m_2} y \pm \sqrt{n_1 n_2} z = 0$$

intersect the conic

$$\begin{aligned}F' &= l_1 l_2 c^2 + m_1 m_2 y^2 + n_1 n_2 z^2 - (m_1 n_2 + m_2 n_1) y z \\ &\quad - (n_1 l_2 + n_2 l_1) z x - (l_1 m_2 + l_2 m_1) = 0.\end{aligned}$$

If $\left(\frac{1}{l_1}, \frac{1}{m_1}, \frac{1}{n_1}\right)$ and $\left(\frac{1}{l_2}, \frac{1}{m_2}, \frac{1}{n_2}\right)$ be respectively the coordinates of two points O_1 and O_2 , then the conic F' passes through the six points in which the lines joining O_1 and O_2 to the vertices of the triangle of reference meet the opposite sides of that triangle; but these six points are the points in which the conics

$$S_3 \equiv \sqrt{l_1 x} + \sqrt{m_1 y} + \sqrt{n_1 z} = 0,$$

$$S_4 \equiv \sqrt{l_2 x} + \sqrt{m_2 y} + \sqrt{n_2 z} = 0$$

touch the sides of the triangle of reference. Hence the conic F'' is the harmonic conic of S_3 and S_4 .

To obtain the equations of the four common tangents of S_1 and S_2 let us form the four conics

$$T_1 \equiv L_1 L_2 - c_1^2 = 0,$$

$$T_2 \equiv L_1 L_2 - c_2^2 = 0,$$

$$T_3 \equiv L_1 L_2 - c_3^2 = 0,$$

$$T_4 \equiv L_1 L_2 - c_4^2 = 0,$$

and write down their isogonal transformations which will give the equations of the common tangents.

Let

$$X_0 \equiv \{\sqrt{(m_1 n_2)} - \sqrt{(m_2 n_1)}\}^2, \quad X_1 \equiv \{\sqrt{(m_1 n_2)} + \sqrt{(m_2 n_1)}\}^2,$$

$$Y_0 \equiv \{\sqrt{(n_1 l_2)} - \sqrt{(n_2 l_1)}\}^2, \quad Y_1 \equiv \{\sqrt{(n_1 l_2)} + \sqrt{(n_2 l_1)}\}^2,$$

$$Z_0 \equiv \{\sqrt{(l_1 m_2)} - \sqrt{(l_2 m_1)}\}^2, \quad Z_1 \equiv \{\sqrt{(l_1 m_2)} + \sqrt{(l_2 m_1)}\}^2,$$

then the equations of the common tangents of S_1 and S_2 are

$$t_1 \equiv X_0 x + Y_0 y + Z_0 z = 0,$$

$$t_2 \equiv X_0 x + Y_1 y + Z_1 z = 0,$$

$$t_3 \equiv X_1 x + Y_0 y + Z_1 z = 0,$$

$$t_4 \equiv X_1 x + Y_1 y + Z_0 z = 0.$$

If we seek to obtain the common tangents of S_1 and S_2 by solving for λ , μ , ν from the equations

$$\lambda x + \mu y + \nu z = 0,$$

$$\sqrt{(l_1 \lambda)} + \sqrt{(m_1 \mu)} + \sqrt{(n_1 \nu)} = 0,$$

$$\sqrt{(l_2 \lambda)} + \sqrt{(m_2 \mu)} + \sqrt{(n_2 \nu)} = 0,$$

we get, on account of the ambiguities of sign in the two equations of condition of tangency, the set of eight lines represented by the equation

$$\begin{aligned} \{\sqrt{(m_1 n_2)} \pm \sqrt{(m_2 n_1)}^2 x\} + \{\sqrt{(n_1 l_2)} \pm \sqrt{(n_2 l_1)}^2 y \\ + \{\sqrt{(l_1 m_2)} \pm \sqrt{(l_2 m_1)}\} z = 0, \end{aligned}$$

of which four are the tangents $t_1 t_2 t_3 t_4$.

The remaining four lines may be written

$$p_1 \equiv X_1 x + Y_1 y + Z_1 z = 0,$$

$$p_2 \equiv X_1 x + Y_0 y + Z_0 z = 0,$$

$$p_3 \equiv X_0 x + Y_1 y + Z_0 z = 0,$$

$$p_4 \equiv X_0 x + Y_0 y + Z_1 z = 0.$$

Let now the equations of the following conics be

$$F' - c_1^2 = 0, \quad F' - c_2^2 = 0, \quad F' - c_3^2 = 0, \quad F' - c_4^2 = 0$$

be formed. They are found to be respectively

$$P_1 \equiv X_1 yz + Y_1 zx + X_1 xy = 0,$$

$$P_2 \equiv X_1 yz + Y_2 zx + Z_6 xy = 0,$$

$$P_3 \equiv X_6 yz + Y_1 zx + Z_6 xy = 0,$$

$$P_4 \equiv X_6 yz + Y_6 zx + Z_6 xy = 0,$$

hence we see that these conics are the isogonal transformations of p_1 , p_2 , p_3 , and p_4 respectively.

It follows that the 16 points found on F may be regarded as the isogonal conjugates of the 16 points in which—four by four—each of the lines $c_1 c_2 c_3 c_4$ is met respectively by the isogonal transformations of the pairs $t_1 p_1$; $t_2 p_2$; $t_3 p_3$; $t_4 p_4$.

The isogonal conjugate of the four points in which c_1 is met by the conics T_1 and P_1 will lie on the conic which is the isogonal transformation of c_1 ; hence the 16 points on F lie four by four on the four conics

$$\sqrt{(l_1 l_2)} yz \pm \sqrt{(m_1 m_2)} zx \pm \sqrt{(n_1 n_2)} xy = 0.$$

It may easily be shown that

$$\begin{aligned} F &= t_1 p_1 + 4 \sqrt{(\Delta_1 \Delta_2)} \{ \sqrt{(l_1 l_2)} yz + \sqrt{(m_1 m_2)} zx + \sqrt{(n_1 n_2)} xy \}, \\ &= t_2 p_2 + 4 \sqrt{(\Delta_1 \Delta_2)} \{ \sqrt{(l_1 l_2)} yz - \sqrt{(m_1 m_2)} zx - \sqrt{(n_1 n_2)} xy \}, \\ &= t_3 p_3 + 4 \sqrt{(\Delta_1 \Delta_2)} \{ -\sqrt{(l_1 l_2)} yz + \sqrt{(m_1 m_2)} zx - \sqrt{(n_1 n_2)} xy \}, \\ &= t_4 p_4 + 4 \sqrt{(\Delta_1 \Delta_2)} \{ -\sqrt{(l_1 l_2)} yz - \sqrt{(m_1 m_2)} zx + \sqrt{(n_1 n_2)} xy \}, \end{aligned}$$

where Δ_1 and Δ_2 are the discriminants of S_1 and S_2 respectively.

It also follows that

$$F = \frac{1}{4} (t_1 p_1 + t_2 p_2 + t_3 p_3 + t_4 p_4) = \frac{1}{4} \Sigma (tp).$$

It will be noticed that F' passes through the intersection of the conics

$$l_1 l_2 x^2 + m_1 m_2 y^2 + n_1 n_2 z^2 = 0,$$

$$(m_1 n_2 + m_2 n_1) yz + (n_1 l_2 + n_2 l_1) zx + (l_1 m_2 + l_2 m_1) xy = 0.$$

The former of these is the 14-point conic of the system of lines

$$\sqrt{(l_1 l_2)} x \pm \sqrt{(m_1 m_2)} y \pm \sqrt{(n_1 n_2)} z = 0,$$

while the latter is the isogonal transformation of the line

$$L \equiv (m_1 n_2 + m_2 n_1) x + (n_1 l_2 + n_2 l_1) y + (l_1 m_2 + l_2 m_1) z = 0.$$

It is at once shown that

$$F = L^2 - 4\Delta_1\Delta_2S_0,$$

where

$$S_0 \equiv \frac{x^2}{l_1l_2} + \frac{y^2}{m_1m_2} + \frac{z^2}{n_1n_2} = 0.$$

This conic S_0 is the 14-point conic of the system of lines

$$\frac{x}{\sqrt{(l_1l_2)}} \pm \frac{y}{\sqrt{(m_1m_2)}} \pm \frac{z}{\sqrt{(n_1n_2)}} = 0.$$

$$\text{If } f_1 \equiv \frac{x}{\sqrt{(l_1l_2)}} + \frac{y}{\sqrt{(m_1m_2)}} + \frac{z}{\sqrt{(n_1n_2)}},$$

$$f_2 \equiv \frac{x}{\sqrt{(l_1l_2)}} - \frac{y}{\sqrt{(m_1m_2)}} - \frac{z}{\sqrt{(n_1n_2)}},$$

$$f_3 \equiv -\frac{x}{\sqrt{(l_1l_2)}} + \frac{y}{\sqrt{(m_1m_2)}} - \frac{z}{\sqrt{(n_1n_2)}},$$

$$f_4 \equiv -\frac{x}{\sqrt{(l_1l_2)}} - \frac{y}{\sqrt{(m_1m_2)}} + \frac{z}{\sqrt{(n_1n_2)}},$$

then we have

$$p_1 - t_1 = 2\sqrt{(\Delta_1\Delta_2)}f_1, \quad p_2 - t_2 = 2\sqrt{(\Delta_1\Delta_2)}f_2,$$

$$p_3 - t_3 = 2\sqrt{(\Delta_1\Delta_2)}f_3, \quad p_4 - t_4 = 2\sqrt{(\Delta_1\Delta_2)}f_4,$$

$$\text{while } p_1 + t_1 = p_2 + t_2 = p_3 + t_3 = p_4 + t_4 = 2L;$$

hence L forms harmonic pencils with each of the triads of lines $p_1t_1f_1$; $p_2t_2f_2$; $p_3t_3f_3$; $p_4t_4f_4$.

We also have

$$\Sigma(t^2) = \Sigma(p^2) = 4\{F + 2\Delta_1\Delta_2S_0\},$$

$$t_1t_2t_3t_4 = F^2 - 4\Delta_1\Delta_2S_1S_2$$

$$= p_1p_2p_3p_4 - 128Lxyz.$$

If we form the equations

$$t_1t_2 - t_3t_4 = 0, \quad t_1t_3 - t_2t_4 = 0, \quad t_1t_4 - t_2t_3 = 0,$$

$$p_1p_3 - p_2p_4 = 0, \quad p_1p_2 - p_3p_4 = 0, \quad p_1p_4 - p_2p_3 = 0,$$

we obtain the series of conics

$$Lx + 2l_1l_2yz = 0, \quad Ly + 2m_1m_2zx = 0, \quad Lz + 2n_1n_2xy = 0,$$

$$Lx - 2l_1l_2yz = 0, \quad Ly - 2m_1m_2zx = 0, \quad Lz - 2n_1n_2xy = 0.$$

The line $L=0$ is thus seen to be of great importance in the geometry of two conics. Its position may be determined as

follows when that of the points (l_1, m_1, n_1) , (l_2, m_2, n_2) is given:—Construct the lines joining (l_1, m_1, n_1) to $(l_2, -m_2, n_2)$ and to $(l_2, m_2, -n_2)$: let these lines cut the sides CA and AB of the triangle of reference in the points G and H respectively; then GH is the line $L = 0$.

It may be noted that the lines L_1 and L_2 are conjugate with respect to all conics inscribed in the standard quadrilateral

$$\sqrt{(l_1 l_2)} x \pm \sqrt{(m_1 m_2)} y \pm \sqrt{(n_1 n_2)} z = 0.$$

Since the equations of the common tangents of S_1 and S_2 are known, we can determine, without ambiguity, the coordinates of the four common points of S_3 and S_4 . The line t_1 touches S_1 and S_2 ; therefore we have

$$\sqrt{(l_1 X_0)} + \sqrt{(m_1 Y_0)} + \sqrt{(n_1 Z_0)} = 0,$$

$$\sqrt{(l_2 X_0)} + \sqrt{(m_2 Y_0)} + \sqrt{(n_2 Z_0)} = 0;$$

showing that the point $(X_0 Y_0 Z_0)$ lies on both S_3 and S_4 ; hence the coordinates of the four points of intersection of these conics are

$$(X_0 Y_0 Z_0), (X_0 Y_1 Z_1), (X_1 Y_0 Z_1), (X_1 Y_1 Z_0).$$

LIST OF TABLES OF $\int_0^x e^{-t^2} dt$ AND CONNECTED INTEGRALS AND FUNCTIONS.

By J. W. L. Glaisher.

THE integrals $\int_0^x e^{-t^2} dt$ and $\int_x^\infty e^{-t^2} dt$, which are connected by the relation

$$\int_0^x e^{-t^2} dt + \int_x^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2},$$

occur in the theory of errors of observation, in the theory of astronomical refraction, the conduction of heat, &c.

In connection with the theory of errors the function is required in the form $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and in connection with astronomical refraction in the form $\int_x^\infty e^{-t^2} dt$, or, more precisely, in the form $e^{x^2} \int_x^\infty e^{-t^2} dt$.

A good many tables of the integrals, or functions differing from them by a multiplier, have been calculated from time to time on account of their important applications in these subjects; but it is noticeable that in several cases the authors of new tables have been unaware of the existence of some of those already calculated, this probably being partly due to the fact that the integrals were required in more than one branch of mathematics. I have therefore thought that it would be useful to form a complete list of all the tables of this kind, so far as they are known to me, and to classify their contents, showing how far they overlap and which may be regarded as superseded.

It will be seen that the tables of Radau, Markoff, and Burgess are very extensive, and that those of the two last are carried to a good many decimal places.

List of books and papers containing tables of the integrals.

The following list contains tables of

$$\int_x^\infty e^{-t^2} dt, \quad e^{x^2} \int_x^\infty e^{-t^2} dt, \quad \int_0^x e^{-t^2} dt, \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

or their logarithms. All the tables included are wholly or partially the results of original calculations.

(1) Kramp, *Analyse des réfractions astronomiques et terrestres* (Strasbourg, 1799).

There are three tables:

Table I. (pp. 197-202) gives $\int_x^\infty e^{-t^2} dt$

from $x=0.00$ to 1.24 at intervals of $.01$ to 8 places,

„	1.25	„	1.50	„	„	9	„
„	1.51	„	1.99	„	„	10	„
„	2.00	„	3.00	„	„	11	„

with differences to the third and fourth orders.

Table II. (pp. 203-206) gives $\log_{10} \int_x^\infty e^{-t^2} dt$

from $x=0.00$ to 3.00 at intervals of $.01$ to 7 places

with differences to the second order.

Table III. (pp. 207–210) gives $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$

from $x=0.00$ to 3.00 at intervals of $.01$ to 7 places with differences to the second order.

(2) F. W. Bessel, *Fundamenta Astronomice* (Regiomonti, 1818).

Table I. (p. 36) gives $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$

from $x=0.00$ to 1.00 at intervals of $.01$ to 7 places with differences to the second order.

Table II. (p. 37) gives the same function, $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$, for argument $\log_{10} x$ from

$\log x=0.00$ to 1.00 at intervals of $.01$ to 7 places with differences to the second order.

Bessel's Table I., is therefore included in Kramp's Table III. It is not an exact reprint, as the last figure is occasionally changed.

(3) A. M. Legendre, *Traité des fonctions elliptiques* (Paris, 1826).

Two tables on pp. 520–521 of vol. ii., give $\int_0^x \left(\log \frac{1}{t}\right)^{-\frac{1}{2}} dt$ from $\left(\log \frac{1}{v}\right)^{\frac{1}{2}} = 0.00$ to 0.50 at intervals of $.01$ to 10 places and from $x=0.80$ to 0.00 „ „ „ with differences in both tables to the sixth order.

Transforming the integral by putting $\left(\log \frac{1}{t}\right)^{\frac{1}{2}} = u$ it becomes $2 \int_\xi^\infty e^{-u^2} du$, where $\xi = \left(\log \frac{1}{x}\right)^{\frac{1}{2}}$;

Legendre's tables therefore give $2 \int_x^\infty e^{-t^2} dt$

from $x=0.00$ to 0.50 at intervals of $.01$ to 10 places and for $x = \left(\log \frac{1}{v}\right)^{\frac{1}{2}}$, where v has the values $0.80, 0.79, \dots, 0.01, 0.00$ to 10 places.

The two tables slightly overlap and cover the whole range of the function from $x=0$ to $x=\infty$.

(4) J. F. Encke, *Berliner astronomisches Jahrbuch für* 1834 (Berlin, 1832).

There are two tables:

Table I. (pp. 305-308) gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x = 0.00$ to 2.00 at intervals of $.01$ to 7 places with differences to the second order.

Table II. (pp. 309-312) gives $\frac{2}{\sqrt{\pi}} \int_0^{\rho x} e^{-t^2} dt$

from $x = 0.00$ to 3.39 at intervals of $.01$ to 5 places,

„ 3.4 „ 5.0 „ .1 „

with first differences.

The quantity $\rho = .476936276\dots$ is given by the equation

$$\frac{2}{\sqrt{\pi}} \int_0^{\rho} = \frac{1}{2}.$$

(5) J. W. L. Glaisher, "On a class of definite integrals, Part II." (*Phil. Mag.*, Ser. IV., vol. 42, pp. 421-436, December, 1871).

On p. 436 there is a table of $\int_x^{\infty} e^{-t^2} dt$

from $x = 3.00$ to 3.50 at intervals of $.01$ to 11 places,

„ 3.51 „ 4.00 „ „ 13 „

„ 4.01 „ 4.50 „ „ 14 „

This is a continuation of Kramp's Table I.

In this paper (p. 432) I mentioned that, calculating $\int_3^{\infty} e^{-t^2} dt$ from Laplace's continued fractions, the values so found differed in the tenth and eleventh figures from the final value in Kramp's Table I.; and, as he calculated his table by differences, I inferred that a portion of his table was probably incorrect in the last two figures.

(6) T. R. v. Oppolzer, *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, vol. ii. (Leipzig, 1880).

Table X. (pp. 587-589) gives $\int_0^x e^{-t^2} dt$

from $x = 0.00$ to 4.50 at intervals of $.01$ to 10 places.

Table XIV. (p. 603) gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x=0.00$ to 2.00 at intervals of $.01$ to 5 places with first differences.

Both these tables were the result of new calculations (see pp. 38 and 297 of the volume). The second table was deduced from the first by multiplication by $\frac{2}{\sqrt{\pi}}$.

(7) R. Radau, "Tables de l'intégrale $\psi(Z) = e^{Z^2} \int_Z^\infty e^{-t^2} dt$," *Annales de l'observatoire de Paris, Mémoires*, vol. 18, Paris, 1885, B.1-B.25.

There are two tables:

The first gives $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$

from $x=-0.120$ to 1.010 at intervals of $.001$ to 7 places with first differences.

The second gives the same function, $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$, for argument $\log x$

from $\log x=0.000$ to 1.000 at intervals of $.001$ to 7 places with first differences.

These extensive tables were the result of a new calculation. Radau found that the last figure in Bessel's Table I. as well as in Kramp's Table III. (portion from $x=0$ to 1) was uncertain to the extent of a unit and that the last figures in Bessel's Table II. (in which the argument is $\log x$) were sometimes in error by 2 or 3 units. He found also that towards the end of Kramp's Table III., the error amounted to as much as 20 in the seventh decimal.

(8) A. Markoff, "Table des valeurs de l'intégrale $\int_x^\infty e^{-t^2} dt$ " (St. Pétersbourg, 1888, pp. xxvii. and 98 pp. of tables).

There are two tables:

The first (pp.1-90) gives $\int_x^\infty e^{-t^2} dt$

from $x=0.000$ to 2.999 at intervals of $.001$ to 11 places,

„ 3.00 „ 4.80 „ „ .01 „ .

Differences are given as far as they are required. They extend to the third order over the greater part of the table.

The second table (pp. 91–98) gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x=0.000$ to 2.499 at intervals of $.001$ to 6 places,

„ 2.50 „ 3.79 „ „ $.01$ „ 7 „

This table is arranged like an ordinary table of logarithms and is without differences.

It would seem that M. Markoff's calculations were suggested by those of M. Radau, for in the preface he refers to the fact that M. Radau had found that in Kramp's Table III. the errors near the end amounted to as much as 20 units in the seventh decimal, and that M. Radau had calculated new tables of the function in question. He states that his own calculations were made in the years 1885–1887 and that he had found the last figure was also uncertain in Kramp's Table I.

(9) B. Kämpfe, “Tafel des Integral $\Phi(\gamma) = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2} dt$ ” (*Wundt's Phil. Stud.*, vol. 9, 1894, pp. 145–150).

The table gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x=0.000$ to 1.509 at intervals of $.001$ to 4 places,

„ 1.50 „ 2.89 „ „ $.01$ „ „

The table is taken from Eucke's Table I., the values for the extra arguments (between 0 and 1.51) being obtained by interpolation.

(10) J. Burgess, “On the definite integral $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$, with extended table of values” (*Trans. Roy. Soc. Edin.*, vol. 39, Part ii., No 9, pp. 1–65, 1898).

There are three tables, the first two being very extensive.

Table I. (pp. 27–39) gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x=0.000$ to 1.249 at intervals of $.001$ to 9 places

with differences to the second order.

Table II. (pp. 40–64) gives the same function $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

from $x=1.000$ to 3.000 at intervals of $.001$ to 15 places

with differences to the fourth order.

Table III. (p. 65) gives $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $\int_x^\infty e^{-t^2} dt$

from $x=3.0$ to $x=5.0$ at intervals of .1 to 15 places and for $x=5.5$ and 6 to 15 places.

In Table I., a column is added giving $\frac{2}{\sqrt{\pi}} e^{-x^2}$ for the same arguments to 9 places, and in Table II. a similar column giving $\log_{10} \frac{2}{\sqrt{\pi}} e^{-x^2}$ for the same arguments to 16 places.

Classified list of tables.

The tables contained in the preceding list may be classified as follows:

Tables of $\int_x^\infty e^{-t^2} dt$.

	x	interval	places
Kramp,	0.00 to 1.24	.01	8,
	1.25 „ 1.50	.01	9,
	1.51 „ 1.99	.01	10,
	2.00 „ 3.00	.01	11.
Glaisher,	3.00 to 3.50	.01	11,
	3.51 „ 4.00	.01	13,
	4.01 „ 4.50	.01	15.
Markoff,	0.000 to 2.999	.001	11,
	3.00 „ 4.80	.01	11.
Burgess,	3.0 to 5.0	.1	15.

Table of $2 \int_x^\infty e^{-t^2} dt$.

	x	interval	places
Legendre,	0.00 to 0.50	.01	10,

and also for $x = \left(\log \frac{1}{v}\right)^{\frac{1}{2}}$, where the values of v are

v	interval	places
0.80 to 0.00	.01	10.

Table of $\log_{10} \int_x^\infty e^{-t^2} dt$.

	x	interval	places
Kramp,	0.00 to 3.00	.01	7.

Tables of $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt.$

		x	interval	places
Kramp,	0.00	to 3.00	.01	7.
Bessel,	0.00	to 1.00	.01	7.
Radau,	-0.120	to 1.010	.001	7.
		$\log x$	interval	places
Bessel,	0.00	to 1.00	.01	7.
Radau,	0.000	to 1.000	.001	7.

Table of $\int_0^x e^{-t^2} dt.$

	x	interval	places
Oppolzer,	0.00 to 4.50	.01	10.

Tables of $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$

	x	interval	places
Encke,	0.00 to 2.00	.01	7.
Oppolzer,	0.00 to 2.00	.01	5.
Markoff,	0.000 to 2.499	.001	6,
	2.50 „ 3.79	.01	7.
Burgess,	0.000 to 1.249	.001	9,
	1.000 „ 3.000	.001	15,
	3.0 „ 5.0	.1	15.
Kämpfe,	0.000 to 1.509	.001	4,
	1.50 „ 2.89	.01	4.

Table of $\frac{2}{\sqrt{\pi}} \int_0^{px} e^{-t^2} dt.$

	x	interval	places
Encke,	0.00 to 3.39	.01	5,
	3.4 „ 5.0	.1	5.

As regards the integral $\int_x^\infty e^{-t^2} dt$ Markoff includes Kramp and the part of Glaisher from 3.00 to 3.50, but Glaisher gives more figures than Markoff over the range 3.51 to 4.50

(interval .01); and to Burgess belongs the range 3.0 to 5.0 (interval .1).

Markoff also includes Legendre, 0.00 to 0.50, the latter values being divided by 2.

As regards the function $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$ Radau includes and supersedes Kramp's table III. (0.00 to 1.00) and Bessel's two tables.

As regards the function $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ Burgess includes all the tables (Encke, Oppolzer, Markoff, Kämpfe).

The tables of $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$ by Kramp, of $\log_{10} e^{x^2} \int_x^\infty e^{-t^2} dt$ by Kramp (1.00 to 3.00), of $\int_0^x e^{-t^2} dt$ by Oppolzer, and of

$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ by Encke have not been recalculated. It is to be noticed however that the last figure in Kramp's tables is uncertain; and that Oppolzer's table may be deduced from Markoff's by subtracting the values in the latter from 0.88622692545....

Comparisons between the Tables.

With two exceptions I have made no comparisons between the different tables. These exceptions are (i) I doubled the values in Markoff's first table for the arguments 0.00 to 0.50 at intervals of .01, thus verifying the first of Legendre's two tables, in which I found only 2 or 3 last figure errors of a unit. (ii) I compared Markoff's table with my own over the range 3.00 to 3.50, the number of figures being 11 in both tables, and found that in 9 cases there was a discrepancy of a unit in the last figure. My last figures could be in error to this extent, and so I have no doubt that Markoff's values are correct. For the range 3.51 to 4.50 my table gives more figures than Markoff's, and it confirms all of his values to the last figure.

It is desirable that a comparison should be made over the common portions of all the tables except in the case of Kramp and Bessel, whose tables may be regarded as superseded.

Values of $\int_x^\infty e^{-t^2} dt$ for $x = 0.5, 1, 1.5, \dots, 4.5$.

Markoff, in the introduction to his tables, has given the values of this integral for $x=0.5$ to 16 places, for $x=1$, 1.5, 2 to 15 places, and for $x=3, 3.5, 4, 4.5$ to 14 places.

The values of the integral for the last four arguments had been given previously by myself and have since been given by Burgess. These values are

x	Glaisher (1871)	Markoff (1888) (figures after the ninth)	Burgess (1898)
3.0	.000 019 577 193,	193 24,	193 237,
3.5	.000 000 658 553 76,	553 78,	553 786,
4.0	.000 000 013 663 189,	663 20,	663 189,
4.5	.000 000 000 174 237 6.	174 24.	174 246.

Thus Markoff's values are all confirmed by Burgess, except for a unit in the last (fourteenth) place.

My value in the case of $x=4.5$ differs from Burgess's by 7 in the fifteenth place; but I do not doubt that his value is correct. As he points out (p. 26 of his paper), his values of Laplace's continued fraction would have enabled him to carry his values of the integral to many more figures.

Markoff was unaware of my calculations, and Burgess of Markoff's.

Reprints of the Tables.

In the list of tables I have made no reference to the reprints of Kramp's and Encke's tables. Reprints of Encke's Table I. are numerous; they frequently occur in works on Probability, sometimes reduced to five figures.

Special mention should be made of De Morgan's reprint of Table I. and II. of Kramp and both tables of Encke at the end (pp. 477-484) of his article, "Theory of Probabilities" in the *Encyclopædia Metropolitana*. In the heading of the former tables occurs the words "From Kramp's *Treatise on Refraction*, corrected." I have accordingly compared Kramp's first two tables with De Morgan's reprint to ascertain the nature of the corrections, and find that there was no recalculation. All the errors corrected in Table I. could be discovered by verifying the formation of the differences, excepting in the case of the value of the integral for argument 2.62, Kramp's value 0.00018716339 being corrected to 0.00018716329. The true value (Markoff) is 0.00018716320. Only one other value of an integral is corrected in this table (a misprint in the value corresponding to argument 1.27), all the other corrections being in the differences. In Table II., which gives the seven-figure logarithms of the values

in Table I., all the corrections in the logarithms (which correspond to the arguments 0.80, 0.81, 0.84, 1.21, 1.22, 1.57, 2.62, 2.74, 2.84) could be discovered from the irregularities in the second differences.

In Fechner's *Elemente der Psychophysik*, vol. i., 1860 (pp. 108, 110), there are small tables giving to four places hD as tabular result to $\frac{r}{n}$ as argument where $\frac{r}{n}$ and hD are connected by the relation

$$\frac{r}{n} = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{1}{2}hD} e^{-t^2} dt.$$

Such tables which are intended for use in statistics are derived by simple processes from the larger tables.

Notation for the integrals.

The integrals $\int_0^x e^{-t^2} dt$ and $\int_x^\infty e^{-t^2} dt$ are of sufficient importance in Analysis (for the expression of other integrals, &c.), even apart from their use in the applications of mathematics, to deserve a recognised notation, and it is a pity that no such notation has arisen. In the *Phil. Mag.* for October, 1871* I denoted $\int_x^\infty e^{-t^2} dt$ by $\text{Erf}x$ and $\int_0^x e^{-t^2} dt$ by $\text{Erfc}x$, these functional signs being abbreviations of the words error-function and error-function-complement. This notation, however, was not well chosen, for it is the latter integral that occurs in the theory of errors, viz. the chance of an error not exceeding x is $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Soon afterwards, therefore, I interchanged the functional symbols, putting

$$\text{Erf}x = \int_0^x e^{-t^2} dt, \quad \text{Erfc}x = \int_x^\infty e^{-t^2} dt,$$

and this notation I used in subsequent papers†.

But, even as altered, the notation is not satisfactory as the error-function should be $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, though $\int_0^x e^{-t^2} dt$ might be called the error-integral and denoted by erix , the

* Ser. IV., vol. 42, pp. 294–302, 421–436.

† As, e.g. in the *Quarterly Journal*, vol. XIII., p. 256 (November, 1874), and the *Messenger*, vol. v., p. 165 (March, 1876).

notation Eix or eix not being available as it had been already occupied by the exponential-integral $\int_0^x \frac{e^{-t}}{t} dt$.

It seems, however, more appropriate that a simple mathematical function should receive its designation and notation from something connected with the function itself or its properties rather from its use in any application of mathematics, so that such a notation as $eiqx = \int_0^x e^{-t^2} dt$ would be preferable, eiq standing for exponential-integral-square, the last word referring to the exponent. The complementary integral $\int_x^\infty e^{-t^2} dt$ might be denoted by $eiq'x$.

A notation for a function does not easily obtain general acceptance unless introduced by the mathematician who first drew attention to it, or gave it its importance, as in the case of the Gamma function, Jacobi's function amx , Weierstrass' P -function, &c., and unfortunately in the case of the functions considered in this paper there is no very substantial authority for any special notation.

As regards the theory of errors, Gauss* used the notation

$$\Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

which was followed by Encke, though not by Oppolzer who used J_x .

As regards the theory of refraction, Kramp† put

$$Ex = e^{x^2} \int_0^x e^{-t^2} dt,$$

and Radau denotes this function by $\psi(x)$, the notation $\Psi(n)$ having been used by Bessel‡ in connection with the function.

If a literal notation derived from the use of the function in applied mathematics be admitted, $refx$ (refraction-function) would not be inappropriate.

* *Werke*, vol. iv., pp. 9, 110.

† *Analyse des Réfractions*, p. 136.

‡ Bessel puts $\Psi(n) = e^{na} \int_{nha^2}^x e^{-t^2} dt$ where $a = \frac{1}{2}\beta \cot^2 \theta$. (*Fundamenta Astronomiæ*, p. 28.)

ON CERTAIN DEFINITE INTEGRALS WHOSE VALUES CAN BE EXPRESSED IN TERMS OF BESSEL'S FUNCTIONS.

By *G. H. Hardy*, Trinity College, Cambridge.

1. It is known that, if a and b are positive, then

$$\begin{aligned}\int_0^\infty \cos\left(ax - \frac{b}{x}\right) \frac{dx}{1+x^2} &= \frac{1}{2}\pi e^{-a-b}, \\ \int_0^\infty \sin\left(ax - \frac{b}{x}\right) \frac{xdx}{1+x^2} &= \frac{1}{2}\pi e^{-a-b}, \\ P \int_0^\infty \cos\left(ax - \frac{b}{x}\right) \frac{dx}{1+x^2} &= \frac{1}{2}\pi \sin(a-b), \\ P \int_0^\infty \sin\left(ax - \frac{b}{x}\right) \frac{xdx}{1-x^2} &= -\frac{1}{2}\pi \cos(a-b),^*\end{aligned}$$

My object in this note is to determine the values of the corresponding integrals in which $ax - (b/x)$ is replaced by $ax + (b/x)$.

I shall suppose $0 < b < a$ and write

$$\tau = \sqrt{b/a}, \quad m = 2\sqrt{ab}.$$

2. The substitution $x = \tau e^u$ gives

$$\begin{aligned}\int_0^\infty \cos\left(ax + \frac{b}{x}\right) \frac{dx}{1+x^2} &= \int_{-\infty}^\infty \frac{\cos(m \cosh u)}{\tau e^u + \tau^{-1} e^{-u}} du \\ &= \int_0^\infty \cos(m \cosh u) \left(\frac{1}{\tau e^u + \tau^{-1} e^{-u}} + \frac{1}{\tau e^{-u} + \tau^{-1} e^{-u}} \right) du \\ &= \frac{\tau^2 + 1}{2\tau} \int_0^\infty \cos(m \cosh u) \frac{\cosh u du}{\theta^2 + \cosh^2 u} \\ &= \frac{a+b}{m} \int_1^\infty \frac{\cos mz}{\sqrt{(z^2-1)}} \frac{z dz}{\theta^2 + z^2},\end{aligned}$$

where $\theta = (1 - \tau^2)/2\tau = (a - b)/m$.

Now consider the complex integral

$$\int \frac{e^{miz}}{\sqrt{(z^2-1)}} \frac{z dz}{\theta^2 + z^2}$$

* See, e.g., *Quarterly Journal*, vol. XXII., p. 374, or Bromwich, *Infinite Series*, p. 493.

taken round a contour formed by (i) the real axis, the points $z = \pm 1$ being avoided by small semicircles lying above it, and (ii) a large semicircle, whose centre is the origin, and which lies above the real axis. Applying Cauchy's theorem in the ordinary manner, we obtain

$$-\int_{-\infty}^{-1} \frac{e^{miz}}{\sqrt{(z^2-1)}} \frac{zdz}{\theta^2+z^2} - i \int_{-1}^1 \frac{e^{miz}}{\sqrt{(1-z^2)}} \frac{zdz}{\theta^2+z^2} \\ + \int_1^{\infty} \frac{e^{miz}}{\sqrt{(z^2-1)}} \frac{zdz}{\theta^2+z^2} = \frac{\pi e^{-m\theta}}{\sqrt{(\theta^2+1)}},$$

the radicals being in every case taken positively.

Hence

$$\int_1^{\infty} \frac{\cos mz}{\sqrt{(z^2-1)}} \frac{zdz}{\theta^2+z^2} = \frac{\pi e^{-m\theta}}{2\sqrt{(\theta^2+1)}} - \frac{1}{2} \int_{-1}^1 \frac{\sin mz}{\sqrt{(1-z^2)}} \frac{zdz}{\theta^2+z^2},$$

$$\text{and } \int_0^{\infty} \cos\left(ax + \frac{b}{x}\right) \frac{dx}{1+x^2} \\ = \frac{1}{2} \pi e^{-(a-b)} - \frac{a+b}{2m} \int_0^{\pi} \sin(m \cos \phi) \frac{\cos \phi d\phi}{\theta^2 + \cos^2 \phi}.$$

$$\text{Now } \frac{\cos \phi}{\theta + \cos^2 \phi} = \frac{4\tau^2 \cos \phi}{1 + 2\tau^2 \cos 2\phi + \tau^4} \\ = \frac{4\tau^2 \cos \phi}{1 - \tau^4} (1 - 2\tau^2 \cos 2\phi + 2\tau^4 \cos 4\phi - \dots) \\ = \frac{4\tau}{1 + \tau^2} (\tau \cos \phi - \tau^3 \cos 3\phi + \tau^5 \cos 5\phi - \dots),$$

$$\sin(m \cos \phi) = 2 \{J_1(m) \cos \phi - J_3(m) \cos 3\phi \dots\};$$

and it is easy to justify the multiplication and subsequent integration term by term of these two series. Hence

$$\int_0^{\pi} \sin(m \cos \phi) \frac{\cos \phi d\phi}{\theta^2 + \cos^2 \phi} = \frac{4\pi\tau}{1 + \tau^2} (\tau J_1 + \tau^3 J_3 + \tau^5 J_5 + \dots).$$

Thus, finally, we obtain

$$(1) \quad \int_0^{\infty} \cos\left(ax + \frac{b}{x}\right) \frac{dx}{1+x^2} \\ = \frac{1}{2} \pi e^{-(a-b)} - \pi (\tau J_1 + \tau^3 J_3 + \tau^5 J_5 + \dots),$$

where $0 < b < a$, $\tau = \sqrt{b/a}$, and the argument of the Bessel's functions is $m = 2\sqrt{ab}$.

3. Similarly, we obtain

$$\begin{aligned}\int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{x dx}{1+x^2} &= \int_{-\infty}^\infty \sin(m \cosh u) \frac{du}{1+\tau^{-2}e^{-2u}} \\ &= \frac{1}{2} \int_1^\infty \frac{\sin mz}{\sqrt{(z^2-1)}} \frac{\tau^2-1+2z^2}{\theta^2+z^2} dz.\end{aligned}$$

By considering the complex integral

$$\int \frac{e^{miz}}{\sqrt{(z^2-1)}} \frac{\tau^2-1+2z^2}{\theta^2+z^2} dz,$$

taken round the same contour as before, we obtain

$$\begin{aligned}\int_1^\infty \frac{\sin mz}{\sqrt{(z^2-1)}} \frac{\tau^2-1+2z^2}{\theta^2+z^2} dz \\ = \frac{2\theta^2+1-\tau^2}{2\theta\sqrt{(\theta^2+1)}} \pi e^{-m\theta} + \frac{1}{2} \int_{-1}^1 \frac{\cos mz}{\sqrt{(1-z^2)}} \frac{\tau^2-1+2z^2}{\theta^2+z^2} dz,\end{aligned}$$

and so

$$\begin{aligned}\int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{x dx}{1+x^2} \\ = \frac{2\theta^2+1-\tau^2}{4\theta\sqrt{(\theta^2+1)}} \pi e^{-m\theta} + \frac{1}{4} \int_{-1}^1 \cos(m \cos \phi) \frac{\tau^2 + \cos 2\phi}{\theta^2 + \cos^2 \phi} d\phi \\ = \frac{1}{2} \pi e^{-(a-b)} + \tau^2 \int_0^1 \cos(m \cos \phi) \frac{\tau^2 + \cos 2\phi}{1+2\tau^2 \cos 2\phi + \tau^4} d\phi.\end{aligned}$$

$$\text{Now } \frac{\tau^2 + \cos 2\phi}{1+2\tau^2 \cos 2\phi + \tau^4} = \cos 2\phi - \tau^2 \cos 4\phi + \tau^4 \cos 6\phi - \dots,$$

$$\cos(m \cos \phi) = J_0(m) - 2J_2(m) \cos 2\phi + 2J_4(m) \cos 4\phi - \dots,$$

and so

$$\begin{aligned}(2) \quad \int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{x dx}{1+x^2} \\ = \frac{1}{2} \pi e^{-(a-b)} - \pi (\tau^2 J_2 + \tau^4 J_4 + \tau^6 J_6 + \dots).\end{aligned}$$

4. A similar process leads to the formulæ

$$\begin{aligned}(3) \quad P \int_0^\infty \cos\left(ax + \frac{b}{x}\right) \frac{dx}{1-x^2} \\ = \frac{1}{2} \pi \sin(a+b) - \frac{1}{2} \int_0^\pi \frac{1-\tau^2}{1-2\tau \cos \phi + \tau^2} \sin(m \cos \phi) d\phi \\ = \frac{1}{2} \pi \sin(a+b) - \pi (\tau J_1 - \tau^3 J_3 + \dots),\end{aligned}$$

$$\begin{aligned}
 (4) \quad P \int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{xdx}{1-x^2} \\
 = -\frac{1}{2}\pi \cos(a+b) + \frac{1}{2} \int_0^\pi \frac{1-\tau^2}{1-2\tau \cos \phi + \tau^2} \cos(m \cos \phi) d\phi \\
 = -\frac{1}{2}\pi \cos(a+b) - \pi(\tau^2 J_2 - \tau^4 J_4 + \dots).
 \end{aligned}$$

5. The formula (2) and (4) may also be deduced from (1) and (3) by means of differentiation with respect to a . Similarly, by differentiation with respect to b , or by an independent investigation, we can deduce the value of

$$\int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{dx}{x(1+x^2)}, \quad P \int_0^\infty \sin\left(ax + \frac{b}{x}\right) \frac{dx}{x(1-x^2)}.$$

We can then, by means of the substitution $x=1/y$, and a subsequent interchange of a and b , determine the values of all these integrals when $0 < a < b$. For example

$$\begin{aligned}
 (1') \quad \int_0^\infty \cos\left(ax + \frac{b}{x}\right) \frac{dx}{1+x^2} \\
 = \frac{1}{2}\pi e^{-(b-a)} - \pi(\tau' J_1 + \tau'^3 J_3 + \dots),
 \end{aligned}$$

where $\tau' = \sqrt{(a/b)}$, if $0 < a < b$.

ON THE ORDER OF THE GROUP OF AUTOMORPHISMS OF AN ABELIAN GROUP.

By Harold Hilton.

1. LET H be an Abelian group of order p^r with a base $[h_1, h_2, h_3, \dots]$ and t_1 invariants $= 1$, t_2 invariants $= 2$, ..., t_r invariants $= r$. Then

$$r = t_1 + 2t_2 + 3t_3 + \dots + rt_r; \text{ let } t = t_1 + t_2 + \dots + t_r.$$

If $[g_1, g_2, g_3, \dots]$ is another base of H , there is an automorphism of H (a simple isomorphism of H with itself) in which g_1 corresponds to h_1 , g_2 to h_2 , g_3 to h_3 , ...; and, conversely, if in any automorphism g_1 corresponds to h_1 , &c. $[g_1, g_2, g_3, \dots]$ is a base of H . Hence the order N of the group of automorphisms (group of isomorphisms) of H = the number of distinct bases of H .

We shall now prove that N is a factor of

$$(p^r - 1)(p^{r-1} - 1) \dots (p - 1)p^{1r(r-1)}$$

and a multiple of $(p - 1)p^{r-1}$.

It is known that $N = f(t_r) \cdot f(t_{r-1}) \dots f(t_1) \cdot p^c$, where $c = \frac{1}{2}t(t-1) + t_1(t-t_1) + t_2(2t-t_2-2t_1) + t_3(3t-t_3-2t_2-3t_1) + \dots$, and $f(k)$ denotes $(p^k-1)(p^{k-1}-1)\dots(p-1)$ while $f(0) = 1$.*

(i) $f(r) \div f(t_r) \cdot f(t_{r-1}) \dots f(t_1)$ is integral. For it

$$= f(r-t) \times \left\{ \frac{f(r)}{f(t_r) \cdot f(r-t_r)} \right\} \times \left\{ \frac{f(r-t_r)}{f(t_{r-1}) \cdot f(r-t_r-t_{r-1})} \right\} \\ \times \left\{ \frac{f(r-t_r-t_{r-1})}{f(t_{r-2}) \cdot f(r-t_r-t_{r-1}-t_{r-2})} \right\} \times \dots \times \left\{ \frac{f(r-t_r-\dots-t_2)}{f(t_1) \cdot f(r-t)} \right\};$$

and each of the expressions in the brackets $\{ \}$ is integral. In fact,

$$\frac{f(r)}{f(t_r) \cdot f(r-t_r)}$$

is the number of subgroups of order p^{t_r} in an Abelian group of order p^r and type $(1, 1, \dots, 1)$, &c.

(ii) It remains to prove that $\frac{1}{2}r(r-1) \geq c \geq r-1$. We can easily verify this for values of r less than 7. Assume it true for every value of r less than the one considered.

If $t = 1$, H is cyclic and $c = r-1$.

If $t = 2$, suppose $t_k = t_{r-k} = 1$ ($2k < r$). Then $c = r + 2k - 2$.

If $t = 2$ and $t_{\frac{1}{2}r} = 2$ (r even); $c = 2r - 3$.

If $t = r-1$, then $t_1 = r-2$, $t_2 = 1$, and $c = \frac{1}{2}r(r-1)$.

If $t = r$, then $t_1 = r$, and $c = \frac{1}{2}r(r-1)$.

Therefore the theorem $\frac{1}{2}r(r-1) \geq c \geq r-1$ is true when $t = 1, 2, r-1$, or r .

If $r-1 > t > 2$, take quantities $\tau, \tau_1, \tau_2, \dots, \tau_r$, such that

$$\tau_1 = t, \tau_2 = t_3, \dots, \tau_{r-1} = t_r, \tau_r = 0, \tau = \tau_1 + \tau_2 + \dots + \tau_r.$$

Then $\tau = t - t_1, \tau + 2\tau_2 + 3\tau_3 + \dots + r\tau_r = r - t$.

Now by the hypothesis, if

$$\gamma = \frac{1}{2}\tau(\tau-1) + \tau_1(\tau-\tau_1) + \tau_2(2\tau-\tau_2-2\tau_1) + \tau_3(3\tau-\tau_3-2\tau_2-3\tau_1) + \dots, \\ \frac{1}{2}(r-t)(r-t-1) \geq \gamma \geq r-t-1;$$

since $0 < r-t < r$.

Now expressing γ in terms of the t 's we have

$$c - \gamma = \frac{1}{2}(2t^2 - t_1^2 - t_1).$$

Therefore

$$\frac{1}{2}[(r-t)(r-t-1) + 2t^2 - t_1^2 - t_1] \geq c \geq r-t-1 + \frac{1}{2}(2t^2 - t_1^2 - t_1),$$

* *Proc. Lond. Math. Soc.* 2, v., top of p. 3; *Annals of Math.* 2, vi., p. 5; *Trans. Amer. Math. Soc.* viii., p. 87.

therefore

$$\frac{1}{2}r(r-1) + \frac{1}{2}[t(3t-2r+1) - t_1^2 - t_1] \geq c \geq (r-1) + \frac{1}{2}(2t^2 - 2t - t_1^2 - t_1).$$

Now

$$t(3t-2r+1) - t_1^2 - t_1 = -\{(t_2 + 3t_3 + 5t_4 + \dots - 1)(t_2 + t_3 + \dots) + 2t_1(t_3 + 2t_4 + 3t_5 + \dots)\};$$

$$\text{and } 2t^2 - 2t - t_1^2 - t_1 = \{(t-t_1)(t+t_1+1) + t(t-3)\}.$$

Remembering that $r-2 > t > 2$, we see at once that each expression in the brackets $\{ \}$ is positive and not zero; so that $\frac{1}{2}r(r-1) > c > r-1$.

Hence the result $\frac{1}{2}r(r-1) \geq c \geq r-1$ can be established by induction.

2. Let H be a normal (self-conjugate) subgroup of a group G of order p^a . Let Γ of order p^β be the subgroup formed by those elements of G which are permutable with every element of H . Since Γ contains H , $\beta \geq r$. It is at once shown that Γ is a normal subgroup of G ; let

$$G \equiv \Gamma g_1 + \Gamma g_2 + \Gamma g_3 + \dots$$

Then no two of the elements $g_1, g_2, g_3, \dots$ transform the elements of H in the same way. Hence G/Γ is a subgroup of the group of automorphisms of H , and it is of order $p^{a-\beta}$. Therefore $\alpha - \beta \leq c$, or $\beta \geq \alpha - c$. If $\alpha > r + c$, $\beta > r$. But $c \leq \frac{1}{2}r(r-1)$; therefore $\beta > r$, if $\alpha > \frac{1}{2}r(r+1)$.

Suppose $\beta > r$, and let g be an element of Γ corresponding to an element of order p common to Γ/H and the central (group of self-conjugate elements) of G/H . Such an element always exists, since Γ/H is a normal subgroup of G/H .* Then $\{H, g\}$ is a normal Abelian subgroup of order p^{r+1} containing H . Hence we have Miller's result†:—

If $\alpha > \frac{1}{2}r(r+1)$, a normal Abelian subgroup of order p^r in a group of order p^a is necessarily contained in a normal Abelian subgroup of order p^{r+1} .‡

If H is cyclic, $c = r-1$. Hence we see, as before, that a normal cyclic subgroup of order p^r in a group of order p^a is contained in a normal Abelian subgroup of order p^{r+1} , provided $\alpha > 2r-1$.

* See, for example, Hilton's *Finite Groups*, p. 143, Ex. 8.

† *Messenger of Mathematics*, vol. XXVII., p. 120; vol. XXXVI., p. 79.

‡ By putting $r=2$ we see that this is not necessarily true when $a = \frac{1}{2}r(r+1)$, unless H has at least two invariants greater than 1.

ON A CLASS OF REGIONS

By J. Mercer, Trinity College, Cambridge.

IN a paper, communicated to the Cambridge Philosophical Society in November last, I quoted the theorem* :—

If R is any region and β a reducible set of points internal to it, then the set of points $(R - \beta)$, which belong to R , but not to β , is a region.

As I have been unable to discover any allusion to this theorem in the memoirs of those writers who have investigated the theory of regions, it seems advisable to give a formal proof of one which includes this as a particular case. For this purpose we shall require two elementary lemmas.

§ 1. Many of the terms which are applied to triangles and sets of triangles may be applied to regions and sets of regions by a slight extension of their use. Thus we shall say that two regions *overlap* if there is one point (and therefore an infinite number of points) internal to each of them. Again a countable set of regions will be said to be *transitive* when at least one of the members of any component overlaps with a region belonging to the complementary component. And lastly, just as in the case of triangles, we shall say that a set of regions *tiles over* the set consisting of those points of the plane which are internal to one or other of the regions.

Dr. Young† has shown that the interior of any region may be tiled over by a set of primitive triangles whose boundaries are points internal to the region. In what follows we shall describe this set of primitive triangles as internal to the region; we shall thus be able to distinguish the set from those which contain primitive triangles having boundary points belonging to the boundary of the region‡.

§ 2. LEMMA I. *The sets of points tiled over by any transitive sets of regions forms a single domain.*

Let $[R]$ § be any transitive set of regions, and $[T]$ the set of all those primitive triangles which belong to one or other of the sets tiling over the interiors of the regions $[R]$. Since $[T]$ is countable and tiles over exactly the same set of points

* *Trans. Camb. Phil. Soc.*, vol. xvi., p. 385.

† The reader who requires information is referred to Chapter IX. of Dr. Young's treatise *The Theory of Sets of Points* (Cambridge, 1906).

‡ *Op. cit.*, p. 180, Theorem 1. The reader will observe that the triangles have the property mentioned, although it is not explicitly stated that they have in the enunciation.

§ The symbol $[x]$ is used to denote a set of which a typical member is x .

as $[R]$, the lemma will be proved when we have shown that $[T]$ is transitive. Suppose, if possible, that this is not the case, but that there is a component $[\tau]$ of $[T]$ whose triangles do not overlap any of those belonging to the complementary component $[\tau']$. Any one of the regions R^* , belonging to the set $[R]$, will be such that its primitive triangles are included in $[T]$, and therefore either in $[\tau]$, or in $[\tau']$. If there were some contained in each, we would be able to divide the primitive triangles of R into two sets, according as they belong to $[\tau]$, or to $[\tau']$. These sets would thus have the property that the triangles belonging to the one do not overlap with triangle belonging to the other; and this we know to be impossible from the definition of a region. It follows that the primitive triangles which tile over any one region R must belong entirely either to $[\tau]$, or to $[\tau']$.

This being so, we can divide our set of regions $[R]$ into two components $[\rho]$ and $[\rho']$, whose generating triangles belong respectively to $[\tau]$, $[\tau']$. At least one region belonging to $[\rho]$ overlaps one belonging to $[\rho']$, because $[R]$ is transitive; hence there must be a triangle of the set $[\tau]$ which overlaps one of the set $[\tau']$. Our hypothesis that $[T]$ was intransitive being at variance with this, we conclude that it is an untenable one. The lemma is therefore established.

§ 3. LEMMA. II. *If R is any region and β a finite number of points internal to it, then the set of points $(R - \beta)$, which belong to R , but not to β , is a region.*

In the first place assume that R is a triangle, and suppose that $P_1, P_2, \dots, P_m$ are the points of the set β . The lemma will evidently be established if we can prove that any two points p_1 and p_2 , which are internal to R , but do not belong to β , can be joined† by a finite number of triangles, whose internal points belong to $(R - \beta)$. For this purpose draw any line l through p_1 in such a way that it passes through no point of β ; and let the lines $p_2P_1, p_2P_2, \dots, p_2P_m$ meet it in $Q_1, Q_2, \dots, Q_m$. If we take any point Q on l so as to be interior to R and distinct from $Q_1, Q_2, \dots, Q_m$, it is clear that the path p_1Qp_2 will lie entirely within R and pass through no point of β . Now the path p_2Qp_1 is a closed set of points, and each point of it is internal to one of the set (S) of primitive triangles, whose interior points belong to R , but not to β . It follows from the Heine-Borel theorem that a finite number of the triangles S will contain every point of the path. Since

* If such exist.

† The term is used in the same sense as Dr. Young's, *op. cit.*, p. 183.

these join p_1 and p_2 , the lemma is established for the case of a triangle.

Suppose next that R is any region whatever. Any one of the primitive triangles of this region (say Δ) can only contain a finite number of points belonging to β ; so that even when such of these points as are included in its interior are regarded as boundary points, it still remains a region (say Δ'); moreover, the set $[\Delta']$ of all regions such as Δ' is transitive, because the set $[\Delta]$ is transitive. It follows that the points which are interior to R , and yet do not belong to β , are tiled over by the transitive set of regions $[\Delta']$; hence, by Lemma I., the set $(R - \beta)$ is a region.

§ 4. Let ν be any ordinal number of the first or second class. The general theorem referred to above may be enumerated as follows:—

If the boundary of any region R is b , and β is any set of points belonging to R , whose ν^{th} derived set consists of points of b only, then the set of points $(R - \beta)$, which belong to R , but not to β , is a region.

Consider in the first place the particular case in which $\nu = 1$. Let Δ be one of the primitive triangles internal to R , and let β_Δ be the set of those points of β which are interior to, or lie on the boundary of, Δ ; then because each point of b is external to Δ , the derived set of β_Δ must be evanescent. Therefore, by Weierstrass's theorem, Δ can only contain a finite number of points belonging to β . By the first part of Lemma II., it follows that the points of Δ , which do not belong to β , form a region, say Δ' . Now, just as in the proof of Lemma II., the points, which are interior to R , and yet do not belong to β , are tiled over by the set of regions $[\Delta']$; and this latter set is transitive. We conclude then from Lemma I. that the set $(R - \beta)$ is a region.

We can next prove the theorem for the case $\nu = 2$. Using the same notation, everything is as before, only that now the first derived set of β_Δ consists of a finite number of points. It follows, by the first part of Lemma II., and what has just been established, that the points of Δ , which do not belong to β , form a region Δ' . Applying Lemmas I. and II., we may now complete the proof by the same reasoning as in the case $\nu = 1$.

Proceeding on these lines, we can prove that the theorem is true when ν is any number of the first class. We proceed to prove that it holds for all values of ν by employing the theorem of generalised induction *

* Cf. Baire, *Functions Discontinues* (Paris, 1905), pp. 47–8.

§ 5. To this effect, let us suppose that the theorem of the preceding paragraph is true for all ordinal numbers less than ν ; we may show from this that it is also true for ν . For, if we adopt the notation just explained, it is clear that the ν^{th} derived set of β_{Δ} is evanescent. In the case when ν has a number immediately preceding it, it follows, from Weierstrass's theorem, that the $(\nu-1)^{\text{th}}$ derivative of β_{Δ} consists of a finite number of points. On the other hand, if ν is a limiting number, let it be the limit of a sequence $\nu_1, \nu_2, \dots, \nu_n, \dots$, in which it is supposed that $\nu_1 > 0$.* If the ν_n^{th} derived set of β_{Δ} contained points for every value of n , the deduced set of these closed sets would, by Cantor's theorem, contain at least one point. But this is impossible since the deduced set, which is the ν^{th} derived set of β_{Δ} , is known to be evanescent. It follows that whatever ν may be there exists a number λ less than it, such that the λ^{th} derived set of β_{Δ} consists of a finite number of points. We conclude, from the first part of Lemma II. and the hypothesis we have adopted, that the points of Δ , which do not belong to β form a region, say Δ' . We may now prove that R is a region just as in the preceding cases. Taking this in conjunction with the fact that we have already established the theorem for the cases $\nu=1, \nu=2$, it follows, by the theorem of generalised induction, that it is true always.

As a corollary of this theorem we have that quoted in the introduction: for since β is reducible, we can find a number ν , such that its ν^{th} derived set is evanescent.

THE SUMMATION OF CERTAIN SERIES.

By *J. H. C. Searle, B.Sc., University College, London.*

PROF. A. C. DIXON has given† the sum of the infinite series

$$1 + \frac{\alpha\beta\gamma}{1!\delta\varepsilon} + \frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}{2!\delta(\delta+1)\varepsilon(\varepsilon+1)} + \dots,$$

subject to the two conditions $\beta + \delta = \gamma + \varepsilon = \alpha + 1$, and the convergency condition—real part of $\delta + \varepsilon - \alpha - \beta - \gamma > 0$. He has used the method of contour integration.

It is possible to sum the series

$$1 + \frac{\alpha\beta\gamma}{abc} + \frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}{a(a+1)b(b+1)c(c+1)} + \dots$$

* This proviso is necessary if we suppose that each set is its own derived set of zero order, for then this derivative may be unclosed.

† *Proc. Lon. Math. Soc.*, 1903, p. 284.

to a finite or infinite number of terms, by elementary methods, given two different conditions, viz.,

$$\alpha + \beta + \gamma = a + b + c - 3,$$

and $\alpha\beta + \beta\gamma + \gamma\alpha = ab + bc + ca - 2(a + b + c) + 3.$

The work may also be extended to the case where there are any number of terms in the numerator and denominator, the number of conditions increasing simultaneously.

Notation. Write

$$[\alpha]_n = \alpha(\alpha + 1) \dots (\alpha + n - 1),$$

$$\left[\frac{\alpha\beta\gamma}{abc} \right]_n = \frac{\alpha(\alpha+1) \dots (\alpha+n-1) \beta(\beta+1) \dots (\beta+n-1) \gamma(\gamma+1) \dots (\gamma+n-1)}{a(a+1) \dots (a+n-1) b(b+1) \dots (b+n-1) c(c+1) \dots (c+n-1)}.$$

Thus $F(\alpha, \beta, \gamma, a, b, c, n)$

$$\begin{aligned} &= 1 + \left[\frac{\alpha\beta\gamma}{abc} \right]_1 + \left[\frac{\alpha\beta\gamma}{abc} \right]_2 + \dots + \left[\frac{\alpha\beta\gamma}{abc} \right]_n \\ &= 1 + \left[\frac{\alpha\beta\gamma}{abc} \right]_1 + \dots + \left[\frac{\alpha\beta\gamma}{abc} \right]_{n-1} + \left[\frac{\alpha\beta\gamma}{abc} \right]_n. \end{aligned}$$

Subtract, therefore

$$\begin{aligned} 0 &= 1 + \frac{\alpha\beta\gamma - abc}{abc} \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_1 \left[\frac{(\alpha+1)(\beta+1)(\gamma+1)}{(a+1)(b+1)(c+1)} - 1 \right] \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_2 \left[\frac{(\alpha+2)(\beta+2)(\gamma+2)}{(a+2)(b+2)(c+2)} - 1 \right] + \dots \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_{n-1} \left[\frac{(\alpha+n-1)(\beta+n-1)(\gamma+n-1)}{(a+n-1)(b+n-1)(c+n-1)} - 1 \right] \\ &\quad - \left[\frac{\alpha\beta\gamma}{abc} \right]_n, \\ 0 &= 1 + \frac{\alpha\beta\gamma - abc}{abc} \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_1 [(\alpha\beta\gamma - abc) + \{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\}] \\ &\quad + \{(\alpha + \beta + \gamma) - (a + b + c)\}] \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_2 [(\alpha\beta\gamma - abc) + 2\{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\}] \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_3 [(\alpha\beta\gamma - abc) + 3\{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\}] \\ &\quad + \dots + \left[\frac{\alpha\beta\gamma}{abc} \right]_{n-1} [(\alpha\beta\gamma - abc) + (n-2)\{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\}] \\ &\quad + \left[\frac{\alpha\beta\gamma}{abc} \right]_n [(\alpha\beta\gamma - abc) + (n-1)\{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\}]. \end{aligned}$$

$$\begin{aligned}
& + 2^3 \{(\alpha + \beta + \gamma) - (a + b + c)\} + \dots \\
& + \frac{[\alpha\beta\gamma]_{n-1}}{[abc]_n} (\alpha\beta\gamma - abc) \\
& + (n-1) \{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\} \\
& + (n-1)^2 \{(\alpha + \beta + \gamma) - (a + b + c)\} \\
& - \left[\frac{\alpha\beta\gamma}{abc} \right]_n \dots \dots \dots (I.).
\end{aligned}$$

Thus, given any two of the three conditions

$$\begin{cases} \alpha + \beta + \gamma = a + b + c \\ \alpha\beta + \beta\gamma + \gamma\alpha = ab + bc + ca \\ \alpha\beta\gamma = abc \end{cases},$$

we can at once obtain the sum of a corresponding series. These series will be of different types.

$$\begin{aligned}
A. \text{ Let } & \alpha + \beta + \gamma = a + b + c \\
& \alpha\beta + \beta\gamma + \gamma\alpha = ab + bc + ca
\end{aligned}$$

in equation (I.).

Therefore

$$\begin{aligned}
0 &= 1 + \frac{\alpha\beta\gamma - abc}{abc} + (\alpha\beta\gamma - abc) \left[\frac{[\alpha\beta\gamma]_1}{[abc]_2} \right. \\
& \quad \left. + \frac{[\alpha\beta\gamma]_2}{[abc]_3} + \dots + \frac{[\alpha\beta\gamma]_n}{[abc]_n} \right] - \left[\frac{\alpha\beta\gamma}{abc} \right]_n \\
&= \frac{[abc]_n - [\alpha\beta\gamma]_n}{[abc]_n} + \frac{\alpha\beta\gamma - abc}{abc} \left[1 + \left[\frac{\alpha\beta\gamma}{(a+1)(b+1)(c+1)} \right] \right. \\
& \quad \left. + \left[\frac{\alpha\beta\gamma}{(a+1)(b+1)(c+1)} \right]_2 + \dots + \left[\frac{\alpha\beta\gamma}{(a+1)(b+1)(c+1)} \right]_{n-1} \right] \\
&= 1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n + \frac{\alpha\beta\gamma - abc}{abc} F\{\alpha, \beta, \gamma, a+1, b+1, c+1, n-1\},
\end{aligned}$$

therefore

$$F\{\alpha, \beta, \gamma, a+1, b+1, c+1, n-1\} = \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{1 - \frac{\alpha\beta\gamma}{abc}}.$$

To find the sum of the infinite series, we require the value of

$$L_{n=\infty} \left[\frac{\alpha\beta\gamma}{abc} \right]_n.$$

This may be obtained as follows:—

We have

$$z(z+1)\dots(z+k-1) = \frac{k!}{\Pi(k, z-1)} k^{z-1},$$

where Π is Gauss' function.

Therefore

$$\begin{aligned} & \left[\frac{\alpha\beta\gamma}{abc} \right]_n \\ &= \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)\gamma(\gamma+1)\dots(\gamma+n-1)}{a(a+1)\dots a+n-1)b(b+1)\dots(b+n-1)c(c+1)\dots(c+n-1)} \\ &= \frac{\Pi(n, \alpha-1)\Pi(n, \beta-1)\Pi(n, \gamma-1)}{\Pi(n, \alpha-1)\Pi(n, \beta-1)\Pi(n, \gamma-1)} n^{\alpha+\beta+\gamma-a-b-c} \\ &= \frac{\Pi(n, \alpha-1)\Pi(n, \beta-1)\Pi(n, \gamma-1)}{\Pi(n, \alpha-1)\Pi(n, \beta-1)\Pi(n, \gamma-1)}, \end{aligned}$$

because $\alpha + \beta + \gamma = a + b + c$.

$$\begin{aligned} \text{Therefore } L_{n=\infty} \left[\frac{\alpha\beta\gamma}{abc} \right]_n &= \frac{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\gamma-1)}{\Pi(\alpha-1)\Pi(\beta-1)\Pi(\gamma-1)} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}. \end{aligned}$$

Therefore $F\{\alpha, \beta, \gamma, a+1, b+1, c+1, \infty\}$

$$= \frac{abc}{abc - \alpha\beta\gamma} \left[1 - \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \right].$$

B. Let $\left. \begin{array}{l} \alpha + \beta + \gamma = a + b + c \\ \alpha\beta\gamma = abc \end{array} \right\} \text{ in (I).}$

Therefore

$$\begin{aligned} 0 &= 1 + \{(\alpha\beta + \beta\gamma + \gamma\alpha) - (ab + bc + ca)\} \\ &\times \frac{[\alpha\beta\gamma]_1}{[abc]_2} \left[1 + 2 \left[\frac{(\alpha+1)(\beta+1)(\gamma+1)}{(a+2)(b+2)(c+2)} \right]_1 + \dots \right. \\ &\left. + (n-1) \left[\frac{(\alpha+1)(\beta+1)(\gamma+1)}{(a+2)(b+2)(c+2)} \right]_{n-2} \right] - \left[\frac{\alpha\beta\gamma}{abc} \right]_n. \end{aligned}$$

Therefore, denoting by G the series in [],

$G\{\alpha+1, \beta+1, \gamma+1, a+2, b+2, c+2, n-2\}$

$$= \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{ab + bc + ca - \alpha\beta - \beta\gamma - \gamma\alpha} \frac{[abc]_n}{[\alpha\beta\gamma]_1}.$$

Putting α for $\alpha + 1$, &c., and noting that the above conditions thereby become

$$\left. \begin{aligned} \alpha + \beta + \gamma &= a + b + c \\ (\alpha - 1)(\beta - 1)(\gamma - 1) &= (a - 1)(b - 1)(c - 1) \end{aligned} \right\} \dots (1),$$

we find

$$\begin{aligned} G\{\alpha, \beta, \gamma, a + 1, b + 1, c + 1, n - 1\} \\ &= \frac{1 - \left[\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{(a - 1)(b - 1)(c - 1)} \right]_{n+1}}{\left\{ \begin{aligned} &(\alpha - 1)(b - 1) + (b - 1)(c - 1) + (c - 1)(a - 1) \\ &- (\alpha - 1)(\beta - 1) - (\beta - 1)(\gamma - 1) - (\gamma - 1)(\alpha - 1) \end{aligned} \right\}} \\ &\times \frac{[(a - 1)(b - 1)(c - 1)]_2}{(\alpha - 1)(\beta - 1)(\gamma - 1)}. \end{aligned}$$

Now

$$[\alpha - 1]_{n+1} = (\alpha - 1)\alpha(\alpha + 1)\dots(\alpha + n + 1) = (\alpha - 1)[\alpha]_n,$$

$$\begin{aligned} \text{therefore } &\left[\frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{(a - 1)(b - 1)(c - 1)} \right]_{n+1} \\ &= \frac{(\alpha - 1)(\beta - 1)(\gamma - 1)}{(a - 1)(b - 1)(c - 1)} \left[\frac{\alpha\beta\gamma}{abc} \right]_n \\ &= \left[\frac{\alpha\beta\gamma}{abc} \right]_n \text{ by (1).} \end{aligned}$$

Again the second equation of (1) gives

$$\alpha\beta\gamma - abc = \alpha\beta + \beta\gamma + \gamma\alpha - ab - bc - ca.$$

Thus $G\{\alpha, \beta, \gamma, a + 1, b + 1, c + 1, n - 1\},$

$$= \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{abc - \alpha\beta\gamma} abc = \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{1 - \frac{\alpha\beta\gamma}{abc}}.$$

The sum to ∞ is, as before,

$$\begin{aligned} G\{\alpha, \beta, \gamma, a + 1, b + 1, c + 1, \infty\} \\ &= \frac{1}{1 - \frac{\alpha\beta\gamma}{abc}} \left[1 - \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \right]. \end{aligned}$$

$$C. \text{ Let } \left. \begin{aligned} \alpha\beta + \beta\gamma + \gamma\alpha &= ab + bc + ca \\ \alpha\beta\gamma &= abc \end{aligned} \right\}$$

in equation (I.).

Therefore

$$0 = 1 + (\alpha + \beta + \gamma - a - b - c) \frac{[\alpha\beta\gamma]_1}{[abc]_2} \\ \times \left[1 + 2^2 \left[\frac{(\alpha + 1)(\beta + 1)(\gamma + 1)}{(a + 2)(b + 2)(c + 2)} \right]_1 + \dots \right. \\ \left. + (n - 1)^2 \left[\frac{(\alpha + 1)(\beta + 1)(\gamma + 1)}{(a + 2)(b + 2)(c + 2)} \right]_{n-2} \right] - \left[\frac{\alpha\beta\gamma}{abc} \right]_n.$$

If, therefore, H denote the series in [], we have

$$H\{\alpha + 1, \beta + 1, \gamma + 1, a + 2, b + 2, c + 2, n - 2\} \\ = \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{a + b + c - \alpha - \beta - \gamma} \frac{[abc]_2}{[\alpha\beta\gamma]}.$$

Putting now α for $\alpha + 1$, &c., and noting the corresponding change in the conditions, viz.,

$$(\alpha - 1)(\beta - 1) + (\beta - 1)(\gamma - 1) + (\gamma - 1)(\alpha - 1) \\ = (a - 1)(b - 1) + (b - 1)(c - 1) + (c - 1)(a - 1), \\ (\alpha - 1)(\beta - 1)(\gamma - 1) = (a - 1)(b - 1)(c - 1),$$

which give $a + b + c - \alpha - \beta - \gamma = abc - \alpha\beta\gamma$,

we have, by reasoning similar to that in B ,

$$H\{\alpha, \beta, \gamma, a + 1, b + 1, c + 1, n - 1\} = \frac{1 - \left[\frac{\alpha\beta\gamma}{abc} \right]_n}{1 - \frac{\alpha\beta\gamma}{abc}}.$$

In order to sum to ∞ , we must now have for convergency

$$\alpha + \beta + \gamma < a + b + c,$$

which entails
$$\lim_{n \rightarrow \infty} \left[\frac{\alpha\beta\gamma}{abc} \right]_n = 0.$$

The sum to ∞ is therefore

$$H\{\alpha, \beta, \gamma, a + 1, b + 1, c + 1, \infty\} = \frac{abc}{abc - \alpha\beta\gamma}.$$

The above results may be generalized if at the same time the number of restrictions be increased. Thus the series

$$F(\alpha, \beta, \dots, \kappa, a, b, \dots, k, n) = 1 + \left[\frac{\alpha\beta\dots\kappa}{ab\dots k} \right]_1 + \dots + \left[\frac{\alpha\beta\dots\kappa}{ab\dots k} \right]_n,$$

or a number of similar series, where each [] is multiplied by

a suitable factor, as in the preceding work, may be summed to a finite or infinite number of terms if we have given any $(s-1)$ of the following relations—

$$\begin{aligned}\Sigma\alpha &= \Sigma a, \\ \Sigma\alpha\beta &= \Sigma ab, \\ \Sigma\alpha\beta\gamma &= \Sigma abc, \\ &\dots\dots\dots \\ \Sigma\alpha\beta\dots\kappa &= \Sigma ab\dots k,\end{aligned}$$

among the $2s$ quantities $\alpha, \beta, \dots, \kappa, a, b, \dots, k$.

Put $s=2$, then we have only one condition to satisfy, viz., $\alpha + \beta = a + b$. If in this series we put $b=1$, we reach the hypergeometric series which has been summed to a finite or infinite number of terms by Prof. M. J. M. Hill,* subject to the condition $\alpha + \beta = a + 1$.

If $s=1$ there are no conditions to satisfy, *i.e.*, the series

$$F(\alpha, \alpha, n) = 1 + \frac{\alpha(\alpha+1)}{\alpha(\alpha+1)} + \dots$$

can always be summed to a finite or, if $\alpha < a$, infinite number of terms.

It can, indeed, be shown by exactly the same method that

$$F\{\alpha, \alpha+1, n-1\} = \frac{1 - \left[\frac{\alpha}{a}\right]_n}{1 - \frac{\alpha}{a}}.$$

We have now

$$\text{L}_{n=\infty} \left[\frac{\alpha}{a}\right]_n = \text{L}_{n=\infty} n^{a-\alpha} \frac{\Pi(n, a-1)}{\Pi(n, a-1)} = 0$$

or ∞ according as $\alpha \leq a$.

Therefore, for convergency, $\alpha < a$, and then

$$F\{\alpha, \alpha+1, \infty\} = \frac{a}{a-\alpha} = \frac{1}{1 - \frac{\alpha}{a}}$$

In conclusion, it may not be out of place to draw attention to the curious similarity which exists between the above results for the hypergeometric series and those for its prototype—the geometric series.

My thanks are due to Prof. Hill for suggesting the above work.

* *Proc. Lond. Math. Soc.*

DIOPHANTINE, &c., FACTORISATION OF QUARTANS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[Continuation of the Paper with this Title, pp. 81-104 of this Vol.]

The author is indebted to Mr H. J. Woodall, A.R.C.Sc., for help in reading the Proof-Sheets, and for useful suggestions.

40. Diophantine Factorisation. Let the 4-tan $N = x^4 + y^4$ be written in the form

$$N = (x^2 + Cy^2)^2 + \mu y^2 \cdot (Rx^2 + Sy^2) \dots \dots \dots (140).$$

This will be reducible to one of the 2^{ic} forms

$$\text{i. } (P^2 - Q^2). \quad \text{ii. } (a^2 + b^2). \quad \text{iii. } (c^2 + 2d^2). \quad \text{iv. } (e^2 - 2f^2) \dots \dots (141),$$

in the following ways i., ii., iii., iv., respectively.

i.	$\mu = -1$;	$N = (x^2 + C'y^2)^2 - y^2 \cdot \{2C'x^2 + (C'^2 - 1)y^2\} \dots \dots \dots (141a),$
		$= P^2 - Q^2 \dots \dots \dots (141b),$
if*		$2C'x^2 + (C'^2 - 1)y^2 = z'^2 \dots \dots \dots (141c).$
ii.	$\mu = +1$;	$N = (x^2 + C''y^2)^2 + y^2 \cdot \{-2C''x^2 - (C''^2 - 1)y^2\} \dots \dots (142a),$
		$= a^2 + b^2 \dots \dots \dots (142b),$
if		$-2C''x^2 - (C''^2 - 1)y^2 = z''^2 \dots \dots \dots (142c).$
iii.	$\mu = +2$;	$N = (x^2 + C'''y^2)^2 + 2y^2 \cdot \{-C'''x^2 - \frac{1}{2}(C'''^2 - 1)y^2\} \dots (143a),$
		$= c^2 + 2d^2 \dots \dots \dots (143b),$
if		$-C'''x^2 - \frac{1}{2}(C'''^2 - 1)y^2 = z'''^2 \dots \dots \dots (143c).$
iv.	$\mu = -2$;	$N = (x^2 + C''''y^2)^2 - 2y^2 \cdot \{C''''x^2 + \frac{1}{2}(C''''^2 - 1)y^2\} \dots \dots (144a),$
		$= e^2 - 2f^2 \dots \dots \dots (144b),$
if		$C''''x^2 + \frac{1}{2}(C''''^2 - 1)y^2 = z'''^2 \dots \dots \dots (144c).$

40a. Factorisant, Characteristic. The four 2^{ic} Diophantine Equations (141a-144a) will be styled *Factorisants* from the property that every solution (x, y) thereof yields an (arithmetic) 2^{ic} partition of one of the forms i., ii., iii., iv., and therefore leads (Art. 26) to a factorisable Quartan.

The quantity C', C'', C''', C'''' which characterises the Factorisants, and on which their solvability depends, will be styled the *Characteristic*. The characteristics and corresponding factorisants will be styled of 1st, 2nd, 3rd, 4th *Class*, according as they fall under Cases i., ii., iii., iv. above.

* This Equation has also been given by Prof. Sanjána of Bombay, (in the *Educational Times* for 1902, see Q. 15110), as suitable for factorisation of Quartans: see the author's solution in *Educational Times*, 1902, p. 337, republished in *Mathematical Questions and Solutions* from same, Vol. III, 1902, p. 49.

The four factorisants may be included under the single type

$$Rx^2 + Sy^2 = z^2 \dots\dots\dots(145),$$

where R, S are functions of the characteristic (C).

40b. 2^{ic} Parts. When a Solution (x, y, z) of any of the factorisants (141a-144a) has been obtained, the 2^{ic} "parts" (P, Q), (a, b), (c, d), (e, f) of the *arithmetic* partitions required for factorisation are (see Art. 40) given by

$$\text{i. } P = x^2 + C'y^2, \quad Q = yz' \dots\dots\dots(146\text{-i.}),$$

$$\text{ii. } a \text{ or } b = x^2 + C''y^2, \quad b \text{ or } a = yz'' \dots\dots\dots(146\text{-ii.}),$$

$$\text{iii. } c = x^2 + C'''y^2, \quad d = yz''' \dots\dots\dots(146\text{-iii.}),$$

$$\text{iv. } e = x^2 + C''''y^2, \quad f = yz'''' \dots\dots\dots(146\text{-iv.}).$$

41. Fractional Symbols. To give this Method full advantage, it is necessary now to remove the restriction (of Art. 2) that all symbols should denote integers. In the present application the case of Half-Quartans will—to save separate detail—be included in general under that of Quartans, so that one set of 2^{ic} forms (i., ii., iii., iv. of Art. 40) will suffice for both.

Here—

N odd requires in (x, y), (P, Q), (a, b), (a, b) one odd, one even... (147a),

„ c, e, c, e all odd; d, f, d, f all even (147b),

N even requires $x, y, a, b, a, b, d, f, d, f$ all odd; c, e, c, e all even.. (147c),

„ $P = \frac{1}{2}(2\varpi + 1), Q = \frac{1}{2}(2\kappa + 1)$ both fractions..... (147d).

And it is also useful to admit

$C, R, S, z; P, Q; a, b, a, b; c, d, c, d; e, f, e, f$ may be fractions.. (147e),

but x, y, Cy^2 , should be* integers..... (147f).

42. Equivalence, Reciprocity. Two characteristics C_1, C_2 (of different classes i., ii., iii., iv., Art. 40), and also the factorisants arising from them, viz.

$$R_1x^2 + S_1y^2 = z_1^2 \text{ (from } C_1); \quad R_2x^2 + S_2y^2 = z_2^2 \text{ (from } C_2) \dots\dots\dots(148),$$

will be said to be *equivalent* or *reciprocal* when they lead to the same Quartan (N), viz.

$$\text{Equivalent with same } x, y; \quad \text{Reciprocal with interchanged } x, y \dots\dots\dots(149).$$

* This is not absolutely necessary (though always convenient). If x, y be fractions, they can be made integers by multiplying by the L.C.M. of their denominators: (see Art. 64b, Ex. 1^o).

The necessary and sufficient condition for this is

$$\text{Equivalence (same } x, y); \quad R_1 : R_2 = S_1 : S_2 \dots\dots\dots(149a).$$

$$\text{Reciprocity (interchanged } x, y); \quad R_1 : S_2 = S_1 : R_2 \dots\dots\dots(149b).$$

This single relation (149a or b) carries with it (by aid of the factorisant) in each case

$$z_1^2 : z_2^2 = \text{the corresponding ratio (149a or b)} \dots\dots\dots(149c),$$

and thereby secures the virtual identity of the two factorisants (148) so that they must lead to the *same*, or *interchanged*, x, y .

Each of these conditions will be found to lead to a single relation between C_1, C_2 as follows—

42a. *Equivalence* (same x, y). The condition (149a) reduces in all the classes to

$$C_1 - \frac{1}{C_1} = C_2 - \frac{1}{C_2}; \quad \text{whence } C_1 C_2 = -1 \dots\dots\dots(150),$$

but this relation appears to exist* only between classes (i., ii.) or (iii., iv.), *i.e.*

$$(C', C''), (C''', C'') \text{ are equivalent if } C' C'' = -1, C''' C'' = -1 \dots(150a).$$

42b. *Reciprocity* (interchanged x, y). The condition (149b) reduces in all classes to

$$\left(C_1 - \frac{1}{C_1}\right)\left(C_2 - \frac{1}{C_2}\right) = 4, \quad \text{whence} \dots\dots\dots(151),$$

$$C_1 = \frac{C_2 + 1}{C_2 - 1} \text{ or } -\frac{C_2 - 1}{C_2 + 1}; \quad C_2 = \frac{C_1 + 1}{C_1 - 1} \text{ or } -\frac{C_1 - 1}{C_1 + 1} \dots\dots\dots(151a),$$

but this relation appears to exist* only between one of classes (i., ii.) and one of (iii., iv.), *i.e.*

Either of C', C'' is reciprocal to one of C''', C'' under condition (151)...(152),

43. *Connected Characteristics.* From Art. 42 it follows that the characteristics (C) always coexist in sets of four (say C', C'', C''', C'')—one in each class—such that

$$(C', C''), (C''', C'') \text{ are equivalent pairs} \dots\dots\dots(152a),$$

$$\text{Either of } C', C'' \text{ with one of } C''', C'' \text{ are reciprocal pairs} \dots\dots(152b),$$

so that if any one (say C_1) be given (in any class), then

$$C_1, -1/C_1, (C_1 + 1)/(C_1 - 1), -(C_1 - 1)/(C_1 + 1) \dots\dots(153),$$

are the complete set of four characteristics related as in Art. 42. The factorisants arising from them will be equivalent or reciprocal by pairs, and will therefore all lead to the *same series* of Quartans.

* *i.e.* with real solutions of the factorisants.

The set of four characteristics and factorisants so related will be styled *Connected Characteristics* and *Factorisants*.

For mere factorisation* purposes it is unnecessary to use all four, as all the factorisation that is possible can be done by any one of them.

44. *General Solutions.* The four factorisants may to some extent be treated of together under the single type

$$Rx^2 + Sy^2 = z^2 \dots\dots\dots(154).$$

Two simple cases arise, viz. :

$$\text{Case 1}^\circ. R = \rho^2; \quad \text{Case 2}^\circ. S = \sigma^2 \dots\dots\dots(155).$$

They admit of general solution in two ways in each of these cases in terms of two arbitrary—but mutually prime—integers (t, u).

CASE 1^o. $R = \rho^2$.

$$1^\circ a. \quad \rho x = t^2 - Su^2, \quad y = 2tu, \quad z = t^2 + Su^2 \dots\dots\dots(156a),$$

$$1^\circ b. \quad \rho x = \frac{1}{2}(t^2 - Su^2), \quad y = tu, \quad z = \frac{1}{2}(t^2 + Su^2) \dots\dots\dots(156b).$$

CASE 2^o. $S = \sigma^2$.

$$2^\circ a. \quad x = 2tu, \quad \sigma y = t^2 - Ru^2, \quad z = t^2 + Ru^2 \dots\dots\dots(157a),$$

$$2^\circ b. \quad x = tu, \quad \sigma y = \frac{1}{2}(t^2 - Ru^2), \quad z = \frac{1}{2}(t^2 + Ru^2) \dots\dots\dots(157b).$$

The quantities C, R, S may be conveniently expressed in terms of ρ, σ .

CASE 1^o. $R = \rho^2$.

$$\text{Class i.} \quad C' = \frac{1}{2}\rho^2, \quad R = \rho^2, \quad S = \left(\frac{1}{2}\rho^4 - 1\right) \dots\dots\dots(158a),$$

$$\text{Class ii.} \quad C'' = -\frac{1}{2}\rho^2, \quad R = \rho^2, \quad S = -\left(\frac{1}{2}\rho^4 - 1\right) \dots\dots\dots(158b),$$

$$\text{Class iii.} \quad C''' = -\rho^2, \quad R = \rho^2, \quad S = -\frac{1}{2}(\rho^4 - 1) \dots\dots\dots(158c),$$

$$\text{Class iv.} \quad C^{iv} = \rho^2, \quad R = \rho^2, \quad S = \frac{1}{2}(\rho^4 - 1) \dots\dots\dots(158d).$$

Changing ρ into $1/\rho$ in (158b, d) gives $(C', C''), (C''', C^{iv})$ equivalent pairs, by (150a).

CASE 2^o. $S = \sigma^2$.

$$\text{Class i.} \quad C' = \frac{k^2 + 1}{k^2 - 1}, \quad R = 2C'', \quad S = \left(\frac{2k}{k^2 - 1}\right)^2 = \sigma^2 \dots\dots(159a),$$

$$\text{Class ii.} \quad C'' = -\frac{k^2 - 1}{k^2 + 1}, \quad R = -2C''', \quad S = \left(\frac{2k}{k^2 + 1}\right)^2 = \sigma^2 \dots\dots(159b),$$

$$\text{Class iii.} \quad C''' = -\frac{2k^2 - 1}{2k^2 + 1}, \quad R = -C''', \quad S = \left(\frac{2k}{2k^2 + 1}\right)^2 = \sigma^2 \dots\dots(159c),$$

$$\text{Class iv.} \quad C^{iv} = \frac{2k^2 + 1}{2k^2 - 1}, \quad R = C^{iv}, \quad S = \left(\frac{2k}{2k^2 - 1}\right)^2 = \sigma^2 \dots\dots(159d).$$

Here $(C', C''), (C''', C^{iv})$ are equivalent pairs, by (150a). Also by (151a),—

Taking $\rho = k$; C', C'' of Case 2^o are reciprocal to C^{iv} of Case 1^o.

And, taking $\frac{1}{2}\rho = k$; C''', C^{iv} of Case 2^o are reciprocal to C' of Case 1^o.

* For this purpose Class i. is usually the most convenient as it yields the co-factors (L, M) directly.

45. Serial Solutions. When any one solution (x_0, y_0, z_0) of a factorisant is known, then writing the factorisant in the three forms

$$\begin{array}{l|l|l} 1^\circ. \quad z^2 - Sy^2 = Rx^2 & 2^\circ. \quad z^2 - Rx^2 = Sy^2 & 3^\circ. \quad Rx^2 + Sy^2 = z^2 \dots (160), \\ S \text{ being } +, \& \neq \square & R \text{ being } +, \& \neq \square & RS \text{ being } -, \& \neq -\square \dots (160a), \end{array}$$

two infinite series of solutions (x_r, y_r, z_r) may usually be obtained for the simple cases of

$$1^\circ. \quad x = x_0 \text{ (const.)}; \quad 2^\circ. \quad y = y_0 \text{ (const.)}; \quad 3^\circ. \quad z = z_0 \text{ (const.)} \dots (160b),$$

respectively, when the conditions stated (as to R, S) are satisfied: (but these conditions can *only be satisfied in two* of the three Cases $1^\circ, 2^\circ, 3^\circ$ at once). This is to be done by aid of the solutions (τ, v) —here supposed known—of the associated Pellian-Equation, which may be here styled the *Unit-form*.

$$1^\circ. \quad \tau^2 - Sv^2 = +1; \quad 2^\circ. \quad \tau^2 - Rv^2 = +1; \quad 3^\circ. \quad \tau^2 + Rv^2 = +1 \dots (161).$$

The first application of the unit-form usually gives *two* distinct new solutions in each case of $1^\circ, 2^\circ, 3^\circ$, viz.

$$\begin{array}{l|l|l} 1^\circ. \quad z_1 = \tau z_0 \mp Svy_0 & 2^\circ. \quad z_1 = \tau z_0 \mp Rvx_0 & 3^\circ. \quad x_1 = \tau x_0 \pm Svy_0 \dots (162a), \\ y_1 = vz_0 \mp \tau y_0 & x_1 = vz_0 \mp \tau x_0 & y_1 = Rvx_0 \mp \tau y_0 \dots (162b), \\ x_1 = x_0 & y_1 = y_0 & z_1 = z_0 \dots \dots \dots (162c), \end{array}$$

wherein the same signs are to be used in any one pair of y_1, z_1 in Cases $1^\circ, 2^\circ$, and opposite signs in Case 3° .

From *each* of these solutions (x_1, y_1, z_1) an infinite series of solutions (x_r, y_r, z_r) may now be derived by the following "succession-formulæ"

$$\begin{array}{l|l|l} 1^\circ. \quad z_{r+1} = \tau z_r + Svy_r & 2^\circ. \quad z_{r+1} = \tau z_r + Rvx_r & 3^\circ. \quad x_{r+1} = \tau x_r - Svy_r \dots (163a), \\ y_{r+1} = vz_r + \tau y_r & x_{r+1} = vz_r + \tau x_r & y_{r+1} = Rvx_r + \tau y_r \dots (163b), \\ x_{r+1} = x_r = x_0 & y_{r+1} = y_r = y_0 & z_{r+1} = z_r = z_0 \dots \dots (163c), \end{array}$$

or by the following (which give either series of x_r, y_r, z_r separately)

$$\begin{array}{l|l|l} 1^\circ. \quad z_{r+1} = 2\tau z_r - z_{r-1} & 2^\circ. \quad z_{r+1} = 2\tau z_r - z_{r-1} & 3^\circ. \quad x_{r+1} = 2\tau x_r - x_{r-1} \dots (164a), \\ y_{r+1} = 2\tau y_r - y_{r-1} & x_{r+1} = 2\tau x_r - x_{r-1} & y_{r+1} = 2\tau y_r - y_{r-1} \dots (164b). \end{array}$$

Thus there are usually *two* infinite Series of Solutions, — arising from the double ($\pm$) sign in the formulæ (162a, b, c) for x_1, y_1, z_1 —in each of the possible Cases of $1^\circ, 2^\circ, 3^\circ$. But, if the use of the $-$ sign in these formulæ leads to

$$\begin{array}{l|l|l} 1^\circ. \quad y_1 = vx_0 - \tau y_0 & 2^\circ. \quad x_1 = vz_0 - \tau x_0 & 3^\circ. \quad y_1 = Rvx_0 - \tau y_0 \\ = 0 \text{ or } y_0 & = 0 \text{ or } x_0 & = 0 \text{ or } y_0 \dots \dots \dots (165), \end{array}$$

then the whole of the series depending thereon disappears, leaving only the one infinite series depending on the use of the + sign in the formula.

Solutions formed as above—i.e. with x , y , or z constant throughout the series—will be styled *Serial Solutions*, and the solution (x_0, y_0, z_0) from which they originate will be styled the *Base-Solution*.

45a. *Simple Serial Solutions.* The series obtained by the Rules of Art. 45, with

$$1^\circ. x_r = x_0 = 1; \quad 2^\circ. y_r = y_0 = 1 \dots\dots\dots(166),$$

giving Simple Quartans $N = (1 + y_r^4)$ or $(x_r^4 + 1)$ are among the most interesting: these will be styled *Simple Serial Solutions*.

46. *Unit-forms.* The unit-forms (161), required to aid in the serial solutions of the factorisants, always admit of solution; but, in some cases the solution is known algebraically; much work may often be saved by observing this.

FORM 1^o, of (161). $\tau^2 - Sv^2 = +1$; S is supposed +.

Class i. Here $S = C''^2 - 1$, then $\tau = C'$, $v = 1$, [if $C' > 1$]. (167).

Class ii. Here $S = 1 - C''^2$, (supposed +, so that $C'' < 1$)... (168).

In this latter case the factorisant, divided by C''^2 , becomes

$$(z/C'')^2 - (1/C''^2 - 1)y^2 = -2x^2/C'' \dots\dots\dots(168a),$$

whilst the unit-form becomes

$$\tau^2 - (1/C''^2 - 1)v^2 = +1 \dots\dots\dots(168b),$$

with solution $\tau = 1/C''$, $v = 1 \dots\dots\dots(169).$

47. *Quartan Chains.* When a series of Quartans $(N_1, N_2, \&c., \dots N_r)$ are algebraically resolvable by one and the same process, each into a pair of co-factors, so that $N_r = L_r \cdot M_r$, and when the alternate co-factors (M_r, L_{r+1}) of each successive pair (N_r, N_{r+1}) are so related that

$$M_r = L_{r+1}, \text{ (for all values of } r) \dots\dots\dots(170),$$

the series is called a *Quartan Chain*. The relation (170) gives

$$x_r^4 + y_r^4 \equiv 0 \pmod{M_r}, \quad x_{r+1}^4 + y_{r+1}^4 \equiv 0 \pmod{L_{r+1}} \dots\dots\dots(170a),$$

so that—since $M_r = L_{r+1}$ —

The ratios $x_r/y_r, x_{r+1}/y_{r+1}$ are different roots of $x^4 + y^4 \equiv 0 \pmod{M_r}$. (170b).

Hence—as one member (x or y) of each ratio is arbitrary—it follows that such Chains might—under the single condition (170)—be formed in an infinite variety of ways at each step. Interesting cases arise when

x_r , or y_r , or $x_r \sim y_r$ is constant throughout the Chain.....(171).

These may be called x -Chains, y -Chains, and $(x \sim y)$ -Chains, respectively.

47a. *x-Chains.* Taking $x_r = x_0$ (= constant), the congruence

$$x_0^4 + y^4 \equiv 0 \pmod{M_r = L_{r+1}} \dots\dots\dots (171a),$$

has four incongruous solutions (Art. 12) at each step

say y_r, y_r', y_r'', y_r''' , each $< M_r$,

so that—at each step (say the r^{th})—the known root (y_r) gives rise to *three* incongruous roots, viz.

$$y_{r+1} = \text{any of } y_r', y_r'', y_r''' (< M_r) \dots\dots\dots (172),$$

each of which satisfies the single chain-condition (170).

Similar results arise when y_r is taken constant. Hence, in both the x - and y -chains a triply infinite series arises.

Similar results must ensue with the $(x \sim y)$ chains, but they are not so easily definable.

47b. *Chain properties.* The chain-condition (170) involves—

$$N_{r+1} : N_r = M_{r+1} : L_r, \text{ for all values of } r \dots\dots\dots (173).$$

$$\text{Hence, } \frac{N_2 \cdot N_4 \cdot N_6 \dots N_{2r}}{N_1 \cdot N_3 \cdot N_5 \dots N_{2r-1}} = \frac{M_{2r}}{L_1} \dots\dots\dots (174a),$$

$$\text{whence } \frac{N_2 \cdot N_4 \cdot N_6 \dots N_{2r}}{N_1 \cdot N_3 \cdot N_5 \dots N_{2r-1}} = M_{2r}, \text{ [if } L_1 = 1] \dots\dots\dots (174b).$$

[For further chain-properties see Art. 61].

47c. *Chain Formation.* Serial Solutions of Class i. have the remarkable property of always yielding Quartan Chains with x constant.

By (48) and (141b)

$$M_r = P_r + Q_r = (x_r^2 + C'y_r^2) + y_r z_r \dots\dots\dots (175a),$$

$$L_{r+1} = P_{r+1} - Q_{r+1} = (x_{r+1}^2 + C'y_{r+1}^2) - y_{r+1} z_{r+1} \dots\dots\dots (175b),$$

Hence, by the chain-condition ($L_{r+1} = M_r$),

$$x_{r+1}^2 + C'y_{r+1}^2 - y_{r+1} z_{r+1} = x_r^2 + C'y_r^2 + y_r z_r, \text{ (for all values of } r). \dots (176).$$

47d. *x*-Chain. The condition for a Serial *x*-solution ($x_r = x_0$, a constant) gives

$$y_{r+1}(C'y_{r+1} - z_{r+1}) = y_r(C'y_r + z_r) \dots\dots\dots(177).$$

The factorisant (Art. 45, Form 1°) may be written

$$z_r^2 - (C'^2 - 1)y_r^2 = 2C'x_0^2 \dots\dots\dots(178),$$

and its "Unit-form" (167) has $\tau = C'$, $v = 1$,

whence the serial succession-formulæ (Art. 45) give

$$y_{r+1} = z_r + C'y_r, \quad -y_r = z_{r+1} - C'y_{r+1} \dots\dots\dots(179),$$

and these values satisfy the above *x*-Chain condition (177) identically. Hence any serial solution (163-1°) of Art. 45, with $x_r = x_0$ (a const.), yields an *x*-Chain in Class i., the only condition being

$$C^2 - 1 \text{ should be } +, \text{ and } \neq \square \dots\dots\dots(179a).$$

The succession-formulæ (163a, b), (164a, b), become

$$z_{r+1} = C'z_r + (C'^2 - 1)y_r, \quad y_{r+1} = z_r + C'y_r \dots\dots\dots(179b),$$

$$z_{r+1} = 2C'z_r - z_{r-1}, \quad y_{r+1} = 2C'y_r - y_{r-1} \dots\dots\dots(179c),$$

and the factors of N_r reduce to

$$L_r = \pm x_0^2 + y_{r-1}y_r, \quad M_r = \pm x_0^2 + y_r y_{r+1} \dots\dots\dots(179d),$$

wherein x_0^2 has the same sign as C' , (if y_{r-1} , y_r , y_{r+1} be taken +).

47e. *y*-Chains. The condition for a Serial *y*-Solution is $y_r = y_0$ (constant). This reduces the chain-condition (176) to

$$x_{r+1}^2 - x_r^2 = y_0(z_{r+1} + z_r) \dots\dots\dots(180).$$

But, from (141c), $z_{r+1}^2 - z_r^2 = 2C'(x_{r+1}^2 - x_r^2)$.

Hence, $z_{r+1} - z_r = 2C'y_0$ (a const.) $\dots\dots\dots(180a)$,

so that the z_r are in arithmetical progression; but this condition seems difficult to realise.

48. *Suitable Characteristics.* To use this Diophantine process successfully the factorisants must be *solvable*. A method will now be developed of determining Characteristics (C) which shall yield solvable factorisants. These will be styled *Suitable Characteristics*. It will be shown that every Quartan will serve to yield *at least nine* suitable characteristics for each (independent) group of four 2^{10} partitions of type (141) in which it can be expressed. The Quartan (N_0) used for this purpose will be styled the *Base-Quartan*.

Let $N_0 = x_0^4 + y_0^4 \dots\dots\dots(181),$

and let $N_0 = P_0^2 - Q_0^2 = a_0^2 + b_0^2 = c_0^2 + 2d_0^2 = e_0^2 - 2f_0^2 \dots\dots\dots(181a),$

be any one set of the 2^{ic} forms of which it is capable.

Then, by (146-i. to iv.)

i. $\pm P_0 = x_0^2 + C'y_0^2, \quad Q_0 = y_0 z_0' \dots\dots\dots(182a),$

whence $C' = (\pm P_0 - x_0^2) / y_0^2 \dots\dots\dots(182b).$

ii. $\pm a_0 \text{ or } \pm b_0 = x_0^2 + C''y_0^2, \quad b_0 \text{ or } a_0 = y_0 z_0'' \dots\dots\dots(183a),$

whence $C'' = (\pm a_0 - x_0^2) / y_0^2, \text{ or } = (\pm b_0 - x_0^2) / y_0^2 \dots\dots(183b).$

iii. $\pm e_0 = x_0^2 + C'''y_0^2, \quad d_0 = y_0 z_0''' \dots\dots\dots(184a),$

whence $C''' = (\pm e_0 - x_0^2) / y_0^2 \dots\dots\dots(184b).$

iv. $\pm e_0 = x_0^2 + C''y_0^2, \quad f_0 = y_0 z_0'' \dots\dots\dots(185a),$

whence $C'' = (\pm e_0 - x_0^2) / y_0^2 \dots\dots\dots(185b).$

Hereby the four 2^{ic} partitions (181a) used above have yielded two values of C' , four of C'' , two of C''' , and two of C'' . And, as x_0, y_0 are interchangeable in the Base-Quartan form (181), and in all the Results following, a second set of values of C', C'', C''', C'' may be evolved.

Hence, in all, four values of C' , eight of C'' , four of C''' , and four of C'' have been found from the single set of four 2^{ic} forms used (181a).

The set of four 2^{ic} forms (181a), used above, may be the algebraic forms (and, in the case of a prime Quartan, they are the only ones available): but any set of the arithmetic forms similar to (181a) of which a composite Quartan N_0 is capable will suffice. And each different set of such forms will yield a different set of Characteristics.

Also all the characteristics (C), found by the above process, will be *suitable*, i.e. they will yield factorisants which are *certainly solvable*, because one solution (viz. the x_0, y_0, z_0 of the Base-Quartan) is known. The total number of factorisants obtainable from each set of four 2^{ic} forms of a given Base-Quartan (N_0) is seen to be 20.

49. *Associates, Primaries & Secondaries.* The set of Characteristics, and their corresponding factorisants, arising from any one set of 2^{ic} forms of a Base-Quartan, will be styled *Associated Characteristics* and *Factorisants* respectively. Those arising from the Algebraic set of 2^{ic} forms will be styled *Primary*; those arising from any arithmetic set of 2^{ic} forms will be styled *Secondary*. Each set contains 20 Associates (Art. 48), but the number of Associates practically

available for factorisation is usually less than 20 for following reasons

1°. Some of the characteristics will be found to be *ineffective* (viz. $C = 0, \pm 1$), in that their factorisants merely reproduce the original 2^{ic} forms instead of new ones(186a).

2°. Some pairs of characteristics will be found to be reciprocal (Art. 42).....(186b).

3°. Some pairs of characteristics may turn out to be *equivalent* (Art. 42)(186c).

49a. Primary Characteristics. The complete set of 20 will be shown (Art. 50, 51) to be virtually reducible to only 10 effective and independent in general (only 9 when $x_0 = 1$), as follows:—

Ineffective, 6; Two C'' , two C''' , two C'''' , [Art. 50](187a).

Reciprocal Pairs, 4; Four C'' with two C''' & two C'''' , [Art. 51]..(187b).

Equivalent Pairs, 1; [when $x_0 = 1$]; C' with C'' , [Art. 51](187c).

Hereby it is seen to be unnecessary to use the primary C''' , C'''' at all, as the only effective ones are reciprocal to some of the C'' .

50. Ineffective Factorisants. It becomes necessary here to introduce notation shewing clearly the effect of the interchange of (x_0, y_0) , and also of (a_0, b_0) ; thus let—

$$N_0 = x_0^4 + y_0^4 = h^4 + k^4 = k^4 + h^4 \dots\dots\dots(188),$$

$$\text{so that} \quad x_0 = h \text{ or } k, \quad y_0 = k \text{ or } h \dots\dots\dots(188a),$$

whilst—(independently of the above interchange of x_0, y_0)—the *algebraic* a_0, c_0, e_0 have the double values—

$$a_0 = \pm h^2, \text{ or } \pm k^2; \quad c_0 = \pm (h^2 - k^2); \quad e_0 = \pm (h^2 + k^2) \dots\dots\dots(188b),$$

whence, for the values of a_0, c_0, e_0, x_0, y_0 stated below, the following values of (the primary) C'' , C''' , C'''' are obtained—

	a_0, c_0, e_0	x_0, y_0	C	a_0, c_0, e_0	x_0, y_0	C
ii.	$a_0 = h^2$	h, k	$C'' = 0$	$a_0 = k^2$	k, h	$C'' = 0 \dots(188c),$
iii.	$c_0 = h^2 - k^2$	h, k	$C''' = -1$	$c_0 = k^2 - h^2$	k, h	$C''' = -1.(188d),$
iv.	$e_0 = h^2 + k^2$	h, k	$C'''' = +1$	$e_0 = -h^2 - k^2$	k, h	$C'''' = +1.(188e),$

and these characteristics lead to the degraded factorisants

$$C'' = 0 \text{ gives } y^2 = z^2; \quad C''' = -1 \text{ and } C'''' = +1 \text{ both give } x^2 = z^2 \dots(188f),$$

and these factorisants are *ineffective*, as they yield only the algebraic 2^{ic} forms of Quartans. Hereby six of the *primary* characteristics are seen to yield *ineffective* factorisants.

51. Equivalent and Reciprocal Factorisants. The following Table shows certain selected (algebraic) values of P_0 , a_0 , c_0 , e_0 , x_0 , y_0 , and the (primary) characteristics (C) deduced therefrom, arranged in pairs in one line.

	x_0, y_0			x_0, y_0	
ii. $a_0 = h^2$	k, h	$C'' = (h^2 - k^2)/h^2$	iii. $c_0 = k^2 - h^2$	h, k	$C''' = (k^2 - 2h^2)/k^2$ (189a),
ii. $a_0 = k^2$	h, k	$C'' = (k^2 - h^2)/k^2$	iii. $c_0 = h^2 - k^2$	k, h	$C''' = (h^2 - 2k^2)/h^2$ (189b),
ii. $a_0 = -h^2$	k, h	$C'' = -(h^2 + k^2)/h^2$	iv. $e_0 = -h^2 - k^2$	h, k	$C''^v = -(2h^2 + k^2)/k^2$ (189c),
ii. $a_0 = -k^2$	h, h	$C'' = -(h^2 + k^2)/k^2$	iv. $e_0 = -h^2 - k^2$	k, h	$C''^v = -(h^2 + 2k^2)/h^2$ (189d),
ii. $a_0 = -1$ [if $h = 1$]	$1, k$	$C'' = -2/k^2$	i. $P_0 = \frac{1}{2}k^4 + 1$ if $h = 1$	$1, k$	$C' = \frac{1}{2}k^2$ (189e).

The pairs (C'' , C'''), (C' , C''^v) in the same line will be found to satisfy the condition (151) of reciprocity, and the last pair (C' , C'') satisfies the condition (150) of equivalence (when $x_0 = h = 1$). Hence among the primary factorisants, there are always *four reciprocal pairs*, and *one equivalent pair* (the latter only when $x_0 = h = 1$). As such pairs yield the same series of Quartans, this *reduces the number* of independent primary factorisants *by four* in all cases, and *by one more* when $x_0 = h = 1$.

51a. Secondary Characteristics. As these depend on arithmetical peculiarities of each *Base-Quartan*, it does not seem possible to trace out the number of Associated Characteristics which are actually effective and independent in any general manner. But, by actual examination of a few Bases, it seems probable that there are

Reciprocal Pairs, 4; viz. (2 of C'' , & C'''), (2 of C'' & C''^v).....(190),
(see the Example $N_0 = 1921$ at end of Art. 54a).

52. Simple General Case. One of the simplest cases, *completely expressible algebraically*, arises from the algebraic 2^{ic} forms of the *Simple Quartan*.

$$N_0 = 1^4 + y_0^4 = (\frac{1}{2}y_0^4 + 1)^2 - (\frac{1}{2}y_0^4)^2 = 1^2 + (y_0^2)^2 \dots \dots \dots (191).$$

These give one pair of *equivalent* values of C' , C'' (Art. 51), viz.

$$C' = \frac{1}{2}y_0^2, \quad C'' = -2/y_0^2 \dots \dots \dots (192).$$

As these are of the forms (158a, b) of Art. 44, the general solutions of their two factorisants may be written down at once by formulæ (156a) of that Article: but the Serial solutions (with $x = 1$ throughout) are more interesting.

The two factorisants given by C' , C'' —(Nos. i., ii. of

Art. 40), written in the form 1° of Art. 45 (suited to serial solution with $x = \text{constant}$) are—

i. $z^2 - (\frac{1}{4}y_0^4 - 1)y^2 = y_0^2.1^2$; ii. $(\frac{1}{2}y_0^2z)^2 - (\frac{1}{4}y_0^4 - 1)y^2 = y_0^2.1^2 \dots (193)$,
and have the same unit-form

$$\tau^2 - Sv^2 = 1, \text{ whence } \tau = \frac{1}{2}y_0^2, \quad v = 1 \dots \dots \dots (193a),$$

and the base-solutions are

$$\text{i. } x_0, y_0, z_0 = 1, y_0, \frac{1}{2}y_0^3; \quad \text{ii. } x_0, y_0, z_0 = 1, y_0, y_0 \dots \dots \dots (193b).$$

They are visibly equivalent (as they differ only in their z , and lead to the same x, y), so that the solutions ($x_r = 1, y_r, z_r$) of either will satisfy the other, provided the z_r be suitably modified, thus—

$$z_r \text{ of Eq. i. is the same as } \frac{1}{2}y_0^2z_r \text{ of Eq. ii.} \dots \dots \dots (193c).$$

Hence, in what follows, the solutions ($x_r = 1, y_r, z_r$) given are only those of Eq. i. of (193). Here Eq. i. gives rise to only one infinite series of x_r, y_r, z_r , because $vz_0 - \tau y_0 = 1$ (Art. 45).

The succession-formulæ of Eq. i. are, by (164a, b),

$$z_{r+1} = y_0^2 z_r - z_{r-1}, \quad y_{r+1} = y_0^2 y_r - y_{r-1} \dots \dots \dots (193d),$$

$$L_r = 1 + y_{r-1} y_r, \quad M_r = 1 + y_r y_{r+1} \dots \dots \dots (194).$$

And these always give rise to a Single Quartan x -Chain, since $M_r = L_{r+1}$ for all values of r (Art. 47).

The succession-formulæ for the three 2^{ic} partitions of the co-factors L_r, M_r where

$$L_r = \alpha_r^2 + \beta_r^2 = \gamma_r^2 + 2\delta_r^2 = \varepsilon_r^2 - 2\phi_r^2 \dots \dots \dots (194a),$$

and the factors are—by (179d)—

$$M_r = \alpha_r'^2 + \beta_r'^2 = \gamma_r'^2 + 2\delta_r'^2 = \varepsilon_r'^2 - 2\phi_r'^2 \dots \dots \dots (194b),$$

are given by

$$\alpha_{r+1} \text{ or } \beta_{r+1} = y_r / y_0 = \alpha'_r \text{ or } \beta'_r; \quad \beta_{r+1} \text{ or } \alpha_{r+1} = y_{r+1} / y_0 = \beta'_r \text{ or } \alpha'_r \dots \dots \dots (195a),$$

$$\gamma_{r+1} = y_0 y_r - \gamma_r = y_0 y_r - \gamma'_{r-1} = \gamma'_r; \quad \delta_{r+1} = y_r - \delta_r = y_r - \delta'_{r-1} = \delta'_r \dots \dots \dots (195b),$$

$$\varepsilon_{r+1} = y_0 y_r + \varepsilon_r = y_0 y_r + \varepsilon'_{r-1} = \varepsilon'_r; \quad \phi_{r+1} = y_r + \phi_r = y_r - \phi'_{r-1} = \phi'_r \dots \dots \dots (195c),$$

and it is hereby seen that each N_r is of the kind studied in Art. 33d (in that $\alpha_r = \alpha'_r$ or $\beta_r = \beta'_r$); but the quantities $\alpha_r, \beta_r, \alpha'_r, \beta'_r$ do not here yield the Quartan Triads of Art. 33a inasmuch as here $m \neq 1$.

It is now easy to write down a few of the values of y_r, z_r , &c, explicitly in terms of y_0 , by the formulæ. *E.g.* those of y_r are

$$r = 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \&c.$$

$$y_r = y_0, \quad y_0^3, \quad (y_0^5 - y_0), \quad (y_0^7 - 2y_0^3), \quad (y_0^9 - 3y_0^5 + y_0), \quad (y_0^{11} - 4y_0^7 + 3y_0^3), \quad \&c \dots (196),$$

and the factors L_r, M_r of N_r can now be written down at once by formulæ (194).

Ex. 1°. Take $N_0 = 1^4 + 2^4 = 17$; whence $C' = 2$,

Factorisant $z^2 - 3y^2 = 4x^2 = 4$, (taking $x_r = x_0 = 1$)(197),

Succession-formulæ, $z_{r+1} = 4z_r - z_{r-1}$, $y_{r+1} = 4y_r - y_{r-1}$ (197a).

The scheme below shows the values of y_r , z_r and the factors* L_r , M_r of $N_r = (1^4 + y_r^4)$ up to $r = 8$: the chain-property ($M_r = L_{r+1}$) will be obvious.

$r = 0,$	$1,$	$2,$	$3,$	$4,$	$5,$	$6,$	$7,$	$8,$	
$y_r =$	2,	8,	30,	112,	418,	1560,	5822,	21728,	81090,
$z_r =$	4,	14,	52,	194,	724,	2702,	10084,	37634,	140452,
$L_r =$	1;	17;	241;	3361;	46817;	652081;	313.29017;	17.89.83609;	7753.227257;
$M_r =$	17;	241;	3361;	46817;	652081;	313.29017;	17.89.83609;	7753.227257;	113.569.381673;

Ex. 2°. Taking $y_0 = 3, 4, 5, 6, 7, 8, 9, 10$, the series of y_r , for each y_0 , may be written down at once from the formulæ (196), and the co-factors* L_r , M_r of each $N_r = (1^4 + y_r^4)$ may be written down at once from the formulæ (194) without the trouble of solving the factorisants each time. The examples below are arranged so as to show the chain property.

x, y	1,3	1,27	1,240	1,2133	1,18957
L	1;	2.41;	6481;	17.30113;	2.17.337.3529;
M	2.41;	6481;	17.30113;	2.17.337.3529;	193.16548577;
x, y	1,4	1,64	1,1020	1,10250	
L	1;	257;	97.673;	449.36929;	
M	257;	97.673;	449.36929;	†	
x, y	1,5	1,125	1,3120	1,77875	
L	1;	2.313;	390001	;	17.113.126481;
M	2.313;	390001;	17.113.126481;	2.	§
x, y	1,6	1,216	1,7770	1,279504	
L	1;	1297;	1678321;	5521.393361;	
M	1297;	1678321;	5521.393361;	§	
x, y	1,7	1,343	1,16800		
L	1;	2.1201;	73.193.409;		
M	2.1201;	73.193.409;	281.49195721;		
x, y	1,8	1,512	1,32760		
L	1;	17.241;	433.38737;		
M	17.241;	433.38737;	†		
x, y	1,9	1,729	1,59040		
L	1;	2.17.193;	97.577.769;		
M	2.17.193;	97.577.769;	‡		
x, y	1,10	1,1000	1,99990		
L	1;	73.137;	99990001;		
M	73.137;	99990001;	†		

* Since $N_r = (1^4 + y_r^4)$, the factors L_r , M_r can be factorised up to very high limits by aid of the Congruence-Tables described in Art. 8-15.

† M has no divisors < 32450 . ‡ M has no divisors < 10000 . § Factors not tried.

52a. *Generalised Formulæ* (of Art. 52). Changing y_r into y_r/x_r and multiplying by such powers of x_r as suffice to remove fractions, and render the results homogeneous, the Simple Quartan takes the general form

$$N_r = x_r^4 + y_r^4, \text{ wherein } x_r = x_0, \dots \dots \dots (198),$$

and y_r is given by—

$$r = 0, 1, 2, 3, 4, \text{ \&c.}$$

$$y_r = y_0, y_0^3, (y_0^5 - x_0^4 y_0), (y_0^7 - 2x_0^4 y_0^3), (y_0^9 - 3x_0^4 y_0^5 + x_0^8 y_0), \text{ \&c.} \dots (198a),$$

and L_r, M_r are given by (194) similarly rendered homogeneous.

$$L_r = x_0^2 + y_{r-1} y_r, \quad M_r = x_0^2 + y_r y_{r+1} \dots \dots \dots (198b).$$

53. *Simplest Base.* The case of $N_0 = 1^4 + 1^4 = 2$, as Base-Quartan, is so exceptional as to merit separate treatment.

Here $x_0 = 1 = y_0$, and $a_0 = 1 = b_0$: hence the interchange of x_0, y_0 , and also that of a_0, b_0 —which usually yield additional cases (Art. 48)—are here alike *ineffective*; from which it results that there are here only two effective and independent characteristics. Thus

$$N_0 = 1^4 + 1^4 = 2 = (\frac{3}{2})^2 - (\frac{1}{2})^2 \text{ gives } P_0 = \pm \frac{3}{2}, Q_0 = \frac{1}{2}, z_0 = \frac{1}{2} \dots \dots (199);$$

$$\text{also, } P_0 = +\frac{3}{2} \text{ gives } C' = +\frac{1}{2}; \quad P_0 = -\frac{3}{2} \text{ gives } C' = -\frac{1}{2} \dots \dots \dots (199a).$$

Examples.

Ex. 1. $C' = +\frac{1}{2}$ gives the factorisant $(2x)^2 - 3y^2 = (2z)^2$, which (Art. 45) yields only a double z -series ($z = z_0 = \frac{1}{2}$, constant), with the unit-form $2^2 - 3.1^2 = +1$. The first few solutions (x_r, y_r), and the co-factors L_r, M_r of each N_r are shown below. [In the z -series 2^0 , the x_r really found are $\frac{1}{2}.7$, $\frac{1}{2}.97$, &c.; the x_r, y_r, z_r printed have all been doubled, so as to obtain integer results (of x, y) throughout].

$$\begin{array}{l} \text{Ser. } 1^0 \\ \left| \begin{array}{l} x, y, z = \\ L = \\ M = \end{array} \right| \begin{array}{l} 1, 1, \frac{1}{2}; \\ 1 \\ 2 \end{array} \left| \begin{array}{l} 13, 15, \frac{1}{2}; \\ 2.137; \\ 17.17 \end{array} \right| \left| \begin{array}{l} 181, 209, \frac{1}{2}; \\ 54497; \\ 2.17.1609; \end{array} \right| \begin{array}{l} \text{Ser. } 2^0 \\ \left| \begin{array}{l} 7, 8, 1; \\ 73; \\ 89; \end{array} \right| \left| \begin{array}{l} 97, 112, 1; \\ 15569; \\ 17.929; \end{array} \right| \left| \begin{array}{l} 1351, 1560, 1 \\ 3035441; \\ 3033561; \end{array} \right| \end{array}$$

Ex. 2. $C' = -\frac{5}{2}$ gives a factorisant which may be written

$$z^2 - \frac{2}{3}.y^2 = -5x^2 \dots \dots \dots (200),$$

which, being of the form 1^0 of Art. 45 is seen to yield a double x -series, with aid of the* unit-form

$$(\frac{5}{2})^2 - 2\frac{1}{3}.1^2 = +1.$$

* The unit-form $55^2 - 21.13^2 = +1$ might have been used, but with far less effect, as the values of y_r which it yields rise rapidly, and it gives no Quartan-Chains. To use this process with the best effect, the unit-form should be chosen with lowest elements (τ_0, v_0).

The succession-formulæ are

$$x_r = x_0 = 1, \quad y_{r+1} = 5y_r - y_{r-1}, \quad z_{r+1} = 5z_r - z_{r-1} \dots \dots \dots (200a),$$

whence may be derived the co-factors of N_r

$$L_r = y_{r-1} y_r - 1, \quad M_r = y_r y_{r+1} - 1 \dots \dots \dots (200b),$$

hence

$$L_{r+1} = M_r \text{ always} \dots \dots \dots (200c),$$

so that each of the series is an x -chain (Art. 47a).

The resulting solutions (x_r, y_r, z_r) are shown in the Table below, with the co-factors L_r, M_r in separate lines so as to exhibit the chain-property clearly.

<i>x-Chain 1°.</i>	$x, y =$	1, 1	1, 2	1, 9	1, 43	1, 206	1, 987	1, 4729	1, 22658
	$z =$	$-\frac{1}{2}$	4	$\frac{1}{2} \cdot 41$	$\frac{1}{2} \cdot 197$	472	$\frac{1}{2} \cdot 4523$	$\frac{1}{2} \cdot 21671$	53916
	$L =$	2;	1;	17;	2.193;	17.521;	203321;	2.41.56921;	89.1203929;
	$M =$	1;	17;	2.193;	17.521;	203321;	2.41.56921;	89.1203929;	*
<i>x-Chain 2°.</i>	$x, y =$	1, 3	1, 14	1, 67	1, 321	1, 1538	1, 7369	1, 35307	
	$z =$	$\frac{1}{2} \cdot 13$	32	$\frac{1}{2} \cdot 307$	$\frac{1}{2} \cdot 1471$	3524	$\frac{1}{2} \cdot 33769$	$\frac{1}{2} \cdot 161797$	
	$L =$	2;	41;	937;	2.10753;	17.113.257;	641.17681;	2.17.7652273;	
	$M =$	41;	937;	2.10753;	17.113.257;	641.17681;	2.17.7652273;		*

54. Characteristics of Simple Quartans. The following Table gives the complete set of 20 (primary) Characteristics of the Simple Base-Quartan

$$N_0 = x_0^4 + y_0^4 = 1^4 + k^4 \dots \dots \dots (201),$$

with the complete set of data

$$x_0, y_0; \quad P_0, Q_0; \quad a_0, b_0; \quad c_0, d_0; \quad e_0, f_0,$$

required for finding them, and the resulting value of z_0 for each; all expressed in terms of k .

Col. headed E, R. The entries here have the following meanings:

The entry *I* denotes that *C* is *ineffective* (Art. 50).

The entry $C_r', C_r'', C_r''', C_r^{iv}$, under *E* or *R*, denotes that the characteristic in the r -th line of class i. ii., iii., iv., respectively, is either *equivalent* or *reciprocal* (Art. 51) to the characteristic in the line in which it falls.

* *M* has no divisors < 10000.

This Table illustrates all the properties of characteristics developed in Art. 48-51.

There is *one* equivalent pair C_1', C_2'' (this property is peculiar to Simple Quartans (here x_0 or $y_0=1$)).

There are *six* ineffective cases (two of C'' , two of C''' , two of C^{iv}); and *four* reciprocal pairs (two of C'', C''' ; two of C'', C^{iv}).

$$N_0 = (1 + k^4).$$

	x_0, y_0	P_0, Q_0	z_0, C'	E, R
i.	1, k	$\frac{1}{2}k^4 + 1, \frac{1}{2}k^4$	$\frac{1}{2}k^3, \frac{1}{2}k^2$	C_2''
	1, k	$-(\frac{1}{2}k^4 + 1), \frac{1}{2}k^4$	$\frac{1}{2}k^3, -(\frac{1}{2}k^4 + 2)/k^2$	
	$k, 1$	$\frac{1}{2}k^4 + 1, \frac{1}{2}k^4$	$\frac{1}{2}k^3, (\frac{1}{2}k^4 - k^2 + 1)$	
	$k, 1$	$-(\frac{1}{2}k^4 + 1), \frac{1}{2}k^4$	$\frac{1}{2}k^3, -(\frac{1}{2}k^4 + k^2 + 1)$	
	x_0, y_0	a_0, b_0	z_0, C''	
ii.	1, k	1, k^2	$k, 0$	$C_1' I$
	1, k	-1, k^2	$k, -2/k^2$	
	$k, 1$	1, k^2	$k^2, 1 - k^2$	
	$k, 1$	-1, k^2	$k^2, -(1 + k^2)$	
	1, k	$k^2, 1$	$1/k, (k^2 - 1)/k^2$	C_3''' C_4^{iv} I
	1, k	- $k^2, 1$	$1/k, -(k^2 + 1)/k^2$	
	$k, 1$	$k^2, 1$	$1, 0$	
	$k, 1$	- $k^2, 1$	$1, -2k^2$	
	x_0, y_0	c_0, d_0	z_0''', C'''	
iii.	1, k	$k^2 - 1, k$	1, $(k^2 - 2)/k^2$	C_3''
	1, k	- $k^2 + 1, k$	1, -1	I
	$k, 1$	$k^2 - 1, k$	$k, -1$	I
	$k, 1$	- $k^2 + 1, k$	$k, (-2k^2 + 1)$	C_5''
	x_0, y_0	e_0, f_0	z^{iv}_0, C^{iv}	
iv.	1, k	$k^2 + 1, k$	1, +1	I
	1, k	$-(k^2 + 1), k$	1, $-(k^2 + 2)/k^2$	C_4''
	$k, 1$	$(k^2 + 1), k$	$k, +1$	I
	$k, 1$	$-(k^2 + 1), k$	$k, -(2k^2 + 1)$	C_6''

54a. *Characteristics, Examples.* The Tables following give the complete sets of 20 characteristics (C', C'', C''', C^{iv}) derived from the small Quartans

$$N_0 = 17; \quad N_0 = 97; \quad N_0 = 1921 = 17.113;$$

together with the complete set of data and results as in Art. 54.

These Tables illustrate all the properties of characteristics developed in Art. 48-51.

$N_0=17$ and 97, (being primes), have only the *primary* system (Art. 49).

$N_0=1921=17.113$, (being composite) has one primary and one secondary system.

$N_0=1^4+2^4$ has *one* equivalent pair (C_1', C_2'' , Art. 51).

Each N_0 has *six* ineffective (two of C' , two of C'' , two of C'''), and *four* reciprocal pairs (two of C'' , C''' , two of C'' , C^{iv}); all in the primary system (Art. 51).

$N_0=1921$ has *one* equivalent pair (C''' , C^{iv}), and *four* reciprocal pairs (two of C'' , C''' ; two of C'' , C^{iv}) in the secondary system; (but it is not known whether this is a *general* property of secondary systems, Art. 51a).

$N_0=17$

$N_0=97$

Ref. N_0	x_0, y_0	P_0, Q_0	z_0, C'	R	x_0, y_0	P_0, Q_0	z_0, C'	R
i. 1	1, 2	9, 8	4, 2	C_2''	3, 2	49, 48	24, 10	
2	1, 2	-9, 8	4, -5/2		3, 2	-49, 48	24, -29/2	
3	2, 1	9, 8	8, 5		2, 3	49, 48	16, 5	
4	2, 1	-9, 8	8, -13		2, 3	-49, 48	16, -53/9	

	x_0, y_0	a_0, b_0	z_0, C''	R	x_0, y_0	a_0, b_0	z_0, C''	R
1	1, 2	1, 4	2, 0	I	3, 2	9, 4	2, 0	I
2	1, 2	-1, 4	2, -1/2	C_1'	3, 2	-9, 4	2, -9/2	
3	2, 1	1, 4	4, -3	C_1'''	2, 3	9, 4	4/3, 5/9	C_2'''
4	2, 1	-1, 4	4, -5	C_2^{iv}	2, 3	-9, 4	4/3, -13/9	C_2^{iv}
ii. 5	1, 2	4, 1	1/2, 3/4	C_3'''	3, 2	4, 9	9/2, -5/4	C_3'''
6	1, 2	-4, 1	1/2, -5/4	C_4^{iv}	3, 2	-4, 9	9/2, -13/4	C_4^{iv}
7	2, 1	4, 1	1, 0	I	2, 3	9, 4	4/3, 0	I
8	2, 1	-4, 1	1, -8		2, 3	-9, 4	4/3, -8/9	

	x_0, y_0	c_0, d_0	z_0, C'''	R	x_0, y_0	c_0, d_0	z_0, C'''	R
1	1, 2	3, 2	1, 1/2	C_3''	3, 2	5, 6	3, -1	I
2	1, 2	-3, 2	1, -1	I	3, 2	-5, 6	3, -7/2	C_3''
3	2, 1	3, 2	2, -1	I	2, 3	5, 6	2, 1/9	C_5''
4	2, 1	-3, 2	2, -7	C_5''	2, 3	-5, 6	2, -1	I

	x_0, y_0	e_0, f_0	z_0, C^{iv}	R	x_0, y_0	e_0, f_0	z_0, C^{iv}	R
1	1, 2	5, 2	1, 1	I	3, 2	13, 6	3, 1	I
2	1, 2	-5, 2	1, -3/2	C_4''	3, 2	-13, 6	3, -11/2	C_4''
3	2, 1	5, 2	2, 1	I	2, 3	13, 6	2, 1	I
4	2, 1	-5, 2	2, -9	C_6''	2, 3	-13, 6	2, -17/9	C_6''

Primary		$N_0=1921=17.113$		Secondary			
	x_0, y_0	P_0, Q_0	z_0', C'	R	P_0, Q_0	z_0', C'	R
i.	5, 6	961, 960	160, 26		65, 48	8, 10/9	
	5, 6	-951, 960	160, -493/18		-65, 48	8, -5/2	
	6, 5	961, 960	192, 37		65, 48	48/5, 29/25	
	6, 5	-961, 960	192, -997/25		-65, 48	48/5, -101/25	
	x, y_0	a_0, b_0	z_0'', C''	R	a_0, b_0	z_0'', C''	R
ii.	5, 6	25, 36	6, 0	I	39, 20	10/3, 7/18	C_3^{iv}
	5, 6	-25, 36	6, -25/18		-39, 20	10/3, -16/9	C_3^{iii}
	6, 5	25, 36	36/5, -11/25	C_1^{iii}	39, 20	4, 3/25	
	6, 5	-25, 36	36/5, -61/25	C_2^{iv}	-39, 20	4, -3	$C_1^{iii}C_2^{iv}$
	5, 6	36, 25	25/6, 11/36	C_4^{iii}	20, 39	13/3, -5/36	
	5, 6	-36, 25	25/6, -61/36	C_4^{iv}	-20, 39	13/3, -5/4	
	6, 5	36, 25	5, 0	I	20, 39	39/5, -16/25	
	6, 5	-36, 25	5, -72/25		-20, 39	39/5, -56/25	
	x_0, y_0	c_0, d_0	z_0''', C'''	R	c_0, d_0	z_0''', C'''	R
iii.	5, 6	11, 30	5, -7/18	C_3''	43, 6	1, 1/2	$C_4''C_2^{iv}$
	5, 6	-11, 30	5, -1	I	-43, 6	1, -17/9	
	6, 5	11, 30	6, -1	I	43, 6	6/5, 7/25	C_2''
	6, 5	-11, 30	6, -47/25	C_5''	-43, 6	6/5, -79/25	
	x_0, y_0	e_0, f_0	z_0^{iv}, C^{iv}	R	e_0, f_0	z_0^{iv}, C^{iv}	R
iv.	5, 6	61, 30	5, +1	I	47, 12	2, 11/18	
	5, 6	-61, 30	5, -43/18	C_4''	-47, 12	2, -2	$C_4''C_1^{iii}$
	6, 5	61, 30	6, +1	I	47, 12	12/5, 11/25	C_1^{iv}
	6, 5	-61, 30	6, -97/25	C_6''	-47, 12	12/5, -83/25	

55. *Examples of Factorisants.* It will now be well to examine one complete set of (primary) factorisants. The number of such factorisants that are both *effective and independent* has been shown to be usually *ten* (Art. 49a), but only *nine* in the case of a Simple Quartan.

The scheme below shows the complete set of 9 for the smallest Base-Quartan $N_0=17$, together with the data from which they arise, viz.

$$(x_0, y_0), (P_0, Q_0), (a_0, b_0); (z_0', C'), (z_0'', C''),$$

brought forward from Table of Art. 54a.

The column headed "Serial" shows the sort of *Serial Solution* possible, i.e. whether with x, y , or z *constant* (Art. 45), thus:—

A single x, y, z in this column shows that only a single x -, y -, or z -series exists; The entry $x, x; y, y; z, z$ indicates that a double x -, y -, or z -series respectively exist (Art. 45).

	x_0, y_0	P_0, Q_0	z_0', C'	Factorisant.	Serial.
i.	1, 2	9, 8	4, 2	$(2x)^2 + 3y^2 = z^2$	x
	1, 2	-9, 8	4, -5/2	$-5x^2 + 21(\frac{1}{2}y)^2 = z^2$	$x, x; z, z$
	2, 1	9, 8	8, 5	$10x^2 + 6(2y)^2 = z^2$	$x, x; y, y$
	2, 1	-9, 8	8, -13	$-26x^2 + 42(2y)^2 = z^2$	$x, x; z, z$
	x_0, y_0	a_0, b_0	z_0'', C''	Factorisant.	Serial.
ii.	2, 1	1, 4	4, -3	$6x^2 - 2(2y)^2 = z^2$	y, z
	2, 1	-1, 4	4, -5	$10x^2 - 6(2y)^2 = z^2$	$y, y; z, z$
	1, 2	4, 1	1/2, 3/4	$-\frac{3}{2}x^2 + 7(\frac{1}{4}y)^2 = z^2$	$x, x; z, z$
	1, 2	-4, 1	1/2, -5/4	$\frac{5}{2}x^2 - (\frac{3}{4}y)^2 = z^2$	$y, y; z, z$
	2, 1	-4, 8	1, -8	$(4x)^2 - 7(3y)^2 = z^2$	z

55a. *Examples, Serial.* Here follow serial solutions (x_r, y_r, z_r) of *seven* of the nine factorisants of the Table of Art. 55, viz. those arising from the characteristics

$$C' = -\frac{5}{2}, 5, -13; \quad C'' = -3, -5, 3/4, -5/4.$$

[The first and last factorisants (with $C' = 2, C'' = -8$) of that Table are here omitted; the solution of the first has been fully given as Ex. 1° in Art. 52; and the latter case is omitted as it yields only z -series].

The head-line of each Example gives—

(1) The reference-number; (2) The characteristic; (3) The factorisant; (4) The unit-form; (5) The value of the constant x or y .

The co factors L_r, M_r are placed in separate lines, so as to shew distinctly the chain-property (this occurs only in the x -series of Class i.).

The x -, and y -, series (*i.e.* with x or y constant) alone are worked* out: those with z constant, being of less interest, are omitted.

In Class i. ($C' = -\frac{5}{2}, 5, -13$) the values of P, Q (from which x, y were obtained) are omitted, as not of much interest. In the x -chains in fact neither z, P, Q are required for mere factorisation of N ; as once the chain is started, the values of L_r, M_r may be (here easily) calculated direct from the x_r, y_r by (194) & (198b).

In Class ii. ($C'' = -3, -5, 3/4, -5/4$), the z^{ic} partitions† of L_r, M_r ,

$$L_r = \alpha_r'^2 + \beta_r'^2, \quad M_r = \alpha_r''^2 + \beta_r''^2, \quad [\text{see (194a, b)}] \dots\dots\dots (202),$$

are given, being the data for computing L_r, M_r .

It will be seen that, where C is fractional, there z, P, Q are also fractional; and when $C' = \frac{1}{2}\omega$, then z, P, Q may each $= \frac{1}{2}\omega$, in which case x, y will be *both odd* and N *even*.

* Note that (to save space), the signs of x, y, z are *omitted*. In using such formulæ as (162a–161b), (179 to 179c) the proper signs must be attached where C is —.

† The accents on α, β , &c., are omitted in the Table to save room.

$$i.-(2). \quad C' = -\frac{1}{2}; \quad z^2 - 24y^2 = -5x^2; \quad (5/2)^2 - 24 \cdot 1^2 = +1; \quad x = x_0 = 1;$$

x -Chain 1°	x, y	1,2	1,9	1,43	1,206	1,987	1,4729	1,22658
	z	8	41	197	944	4523	21671	103832
	L	1;	17;	2,193;	17,521;	203321;	2,41,56921;	89,1203929;
	M	17;	2,193;	17,521;	203321;	2,41,56921;	89,1203929;	*

x -Chain 2°	x, y	1,1	1,3	1,14	1,67	1,321	1,1538	1,7369
	z	1	13	64	307	1471	7048	33769
	L	1;	2;	41;	937;	2,10753;	17,113,257;	641,17681;
	M	2;	41;	937;	2,10753;	17,113,257;	641,17681;	2,17,7652273;

$$i.-(3). \quad C' = 5; \quad z^2 - 24y^2 = 10x^2; \quad 5^2 - 24 \cdot 1^2 = +1; \quad x = x_0 = 2;$$

x -Chain 1°	x, y	2,1	2,13	2,129	2,1277	x -Chain 2°	x, y	2,3	2,31	2,307	2,3039
	z	8	64	632	6256		z	16	152	1504	14888
	L	1;	17;	41,41;	257,641;		L	1;	97;	9521;	17,54881;
	M	17;	41,41;	257,641;	41,393721		M	97;	9521;	17,54481;	91422241;

$$i.-(3). \quad C' = 5; \quad z^2 - 10x^2 = 24y^2; \quad 19^2 - 10 \cdot 6^2 = +1; \quad y = y_0 = 1;$$

y -Series 1°	x, y	2,1	86,1	3266,1	y -Series 2°	x, y	10,1	382,1	14506,1
	z	8	272	10328		z	32	1208	45872
	L	1;	7129;	17,41,15289;		L	73;	17,8513;	210378169;
	M	17;	7673;	10677089;		M	137;	147137;	210469913;

$$i.-(4). \quad C' = -13; \quad z^2 - 168y^2 = -26x^2; \quad 13^2 - 168 \cdot 1^2 = +1; \quad x = x_0 = 2;$$

x -Chain 1°	x, y	2,1	2,21	2,545	2,14149	x -Chain 2°	x, y	2,5	2,129	2,3349
	z	8	272	7064	183392		z	64	1672	43408
	L	1;	17;	17,673;	77,11201;		L	1;	641;	41,41,257;
	M	17;	17,673;	77,11201;	†		M	641;	41,41,257;	4561,63841;

$$ii.-(3). \quad C'' = -3; \quad z^2 - 6x^2 = -8y^2; \quad 5^2 - 6 \cdot 2^2 = +1; \quad y = y_0 = 1;$$

y -Series	x, y	2,1	18,1	178,1	1762,1	17442,1
	z	4	44	436	4316	42724
	α, β	1,0	7,8	73,72	719,720	
	α, β	1,4	23,20	217,220	2159,2156	
	L	1;	113;	10513;	1035361;	§
	M	17;	929;	17,41,137;	73,127529;	§

$$ii.-(4). \quad C'' = -5; \quad z^2 - 10x^2 = -24y^2; \quad 19^2 - 10 \cdot 6^2 = +1; \quad y = y_0 = 1;$$

y -Series 1°	x, y	2,1	62,1	2354,1	89390,1	y -Series 2°	x, y	2,1	14,1	530,1	20126,1
	z	4	196	7444	282676		z	4	44	1676	63644
	α, β	1,0	19,20	745,744			α, β	1,0	5,4	167,168	
	α, β	1,4	101,96	3719,3724			α, β	1,4	19,24	841,836	
	L	1;	761;	1108,561;	§		L	1;	41;	56113;	§
	M	17;	19417;	17,1629361;	§		M	17;	937;	41,34297;	§

* M has no divisors < 10000 . † The composition of this M is not known.

† This factorisation, with its application to factorisation of these Quartans, was given by Prof. Sanjána in *Math. Questions and Solutions from the Educational Times*, Vol. 73, 1906, p. 114 (Solution of Q. 13958).

§ The composition of these large numbers is not known.

ii.—(5). $C'' = 3/4$; $z^2 - \frac{7}{16}y^2 = -\frac{3}{2}x^2$; $\delta^2 - 7.3^2 = +1$; $x = x_0 = 1$;

<i>x</i> -Series 1°.	<i>x, y</i>	1,2	1,22	1,350	1,5578
	<i>2z</i>	1	29	403	7379
	<i>α, β</i>	1,0	3,8	127,48	765,2024
	<i>α, β</i>	1,4	53,20	319,844	13451,5084
	<i>L</i>	1;	73;	18433;	4681801;
	<i>M</i>	17;	3209;	814097;	17.353.34957;
<i>x</i> -Series 2°.		1,2	1,10	1,158	1,2518
		1	13	209	3331
		1,0	3,8	127,48	765,2024
		1,4	11,4	65,172	2741,1036
		1;	73;	18433;	4681801;
		17;	137;	33809;	17.193.26,7;

ii.—(6). $C'' = -5/4$; $x^2 - \frac{5}{2}x^2 = -\frac{9}{16}y^2$; $19^2 - 10.6^2 = +1$; $y = y_0 = 2$;

<i>y</i> -Series 1°.	<i>x, y</i>	1,2	25,2	949,2	36037,2
	<i>2z</i>	1	79	3001	113959
	<i>α, β</i>	1,0	7,8	501,300	
	<i>α, β</i>	1,4	39,44	1501,1496	
	<i>L</i>	1;	113;	313.577;	*
	<i>M</i>	17;	3457;	41.109537;	*
<i>y</i> -Series 2°.		1,2	13,2	493,2	18721,2
		1	41	1559	59201
		1,0	5,4	155,156	
		1,4	21,16	779,784	
		1;	41;	137.353;	*
		17;	17.41;	193.6329;	*

55*b*. 2^{ic} *Partitions of Co-Factors*. It might be expected that the 2^{ic} partitions of the co-factors (*L, M*) of a Serial Solution would be closely connected (195*a, b, c*).

The 2^{ic} partitions of the co-factors (*L, M*) of a selected† number of the Serial solutions of the Tables of Art. 55*a*, viz.:

x-Chains i.—(2), i.—(3); Series ii.—(3), ii.—(4), ii.—(5);

are given in the Table following.

This Table is divided into two halves, left and right: when the Serial solutions occur in pairs (Art. 45), the twin Series are placed one in each half-Table (left and right), and numbered 1° and 2°.

Each series—(when not a Chain)—has *two* bands of four lines each to itself; the upper band for the *L*-factors, the lower band for the *M*-factors. The top line of each band shows the *L*- or *M*-factors brought forward from the Tables of Art. 55*a*. But the Chain-series {i.—(2), i.—(3)} have only a *single* band of four lines each for their *L*-factors; (as here $M_r = L_{r+1}$, so that it is unnecessary to show the *M*-factors). The 2nd, 3rd. and 4th lines of each band show the 2^{ic} partitions (*α, β*), (*γ, δ*), (*ε, φ*) of the *L*- or *M*-factors in the top line.

Notation. In order to bring out clearly the relations of the 2^{ic} partitions, the following notation is adopted,

Twin { Ser. 1°; *L* has (*a'' β' γ' δ' ε' φ'*)_r; *M* has (*a'' β'' γ'' δ'' ε'' φ''*)_r,
Series { Ser. 2°; *L* has (*a''' β''' γ''' δ''' ε''' φ'''*)_r; *M* has (*a'' β'' γ'' δ'' ε'' φ''*)_r.

The subscripts *r*=0, 1, 2, 3, &c., are used only when required to distinguish successive partitions.

* The composition of these large numbers is not known.

† Those of Ser. i.—(3), i.—(4), ii.—(6) have been omitted, as the data are there insufficient to disclose the relation of the 2^{ic} partitions.

i.—(2).	L	x -Chain 1°	1; 17; 2.193; 17.521; 203321;	x -Chain 2°	1; 2; 41; 937; 2.10703;
	α, β		1,0 1.4 5.19 91.24 115.436		1,0 1.1 5.4 19.24 91.115
	γ, δ		1,0 3.2 12.11 55.54 261.260		1,0 0.1 3.4 17.18 84.85
	ϵ, ϕ		1,0 5.2 22.7 107.36 511.170		1,0 2.1 7.2 35.12 166.55
i.—(3).	L	x -Chain 1°	1; 17; 41.41; 257.641;	x -Chain 2°	1; 97; 9021; 17.54881;
	α, β		1,0 1.4 9.40 89.396		1,0 9.4 89.40 881.396
	γ, δ		1,0 3.2 23.24 235.234		1,0 5.6 57.56 557.558
	ϵ, ϕ		1,0 5.2 57.28 573.286		1,0 13.6 137.68 1365.682
ii.—(3).	L	y -Series.	1; 113; 10513; 1035361;		
	α, β		1,0 7.8 73.72 719.720		
	γ, δ		1,0 9.4 89.36 881.360		
	ϵ, ϕ		1,0 11.2 105.16 1043.162		
ii.—(3).	M	y -Series.	17; 929; 17.41.137; 73.127529;		
	α, β		1.4 23.20 217.220 2159.2156		
	γ, δ		3.2 27.10 267.110 2643.1078		
	ϵ, ϕ		5.2 31.4 317.50 3127.484		
ii.—(4).	L	y -Series 1°	1; 761; 1108561;	y -Series 2°	1; 41; 56113;
	α, β		1,0 19.20 745.744		1,0 5.4 167.168
	γ, δ		1,0 27.4 1033.144		1,0 3.4 121.144
	ϵ, ϕ		1,0 31.10 1177.372		1,0 7.2 265.84
ii.—(4).	M	y -Series 1°	17; 19417; 17.1629361;	y -Series 2°	17; 937; 41.34297;
	α, β		1.4 101.96 3719.3724		1.4 19.24 841.836
	γ, δ		3.2 137.18 5163.722		3.2 17.18 603.722
	ϵ, ϕ		5.2 155.48 5885.1862		5.2 35.12 1325.418
iii.—(5).	L	x -Series 1°	1; 73; 18433; 4681801;	x -Series 2°	1; 73; 18433; 4681801;
	α, β		1,0 3.8 127.48 765.2024		1,0 3.8 127.48 765.2024
	γ, δ		1,0 1.6 1.96 1.1530		1,0 1.6 1.96 1.1530
	ϵ, ϕ		1,0 9.2 145.36 2313.578		1,0 9.2 145.36 2313.578
iii.—(5).	M	x -Series 1°	17; 3209; 814097; 17.353.34957;	x -Series 2°	17; 137; 33809; 17.193.2617;
	α, β		1.4 53.20 319.844 13451.5084		1.4 11.4 65.172 2741.1036
	γ, δ		3.2 3.40 3.638 3.10168		3.2 3.8 3.130 3.2072
	ϵ, ϕ		5.2 61.16 965.242 15373.3844		5.2 13.4 197.50 3133.784

The following relations* among the 2^{1c} partitions will be evident from the Table

Twin Chains i.—(2).

$$\alpha'_r + \beta'_r = \alpha'_{r+1} \text{ or } \beta'_{r+1}, (\text{alterny.}); \beta'''_{r+1} - \alpha'''_{r+1} = \alpha'''_r \text{ or } \beta'''_r, (\text{alterny.}). (204a),$$

$$(L'_r = \text{even}), \alpha'_r = \alpha'''_r, \beta'_r = \alpha'''_{r+1}; (L'_r = \text{odd}) \alpha'''_r = \alpha'_{r-1}, \beta'''_r = \alpha'_r \dots (204b),$$

$$\beta'_r = \beta'''_r \text{ or } \beta'''_{r+1} \text{ (if even) } \dots \dots \dots (204c),$$

$$\alpha'_{r+1} \text{ or } \beta'_{r+1} = (-2C')(\alpha'_r \text{ or } \beta'_r) - (\alpha'_{r-1} \text{ or } \beta'_{r-1}) \dots \dots \dots (204d),$$

$$\gamma' - \delta' = +1 = \delta''' - \gamma''' \dots \dots \dots (204e),$$

$$\epsilon' - 3\phi' = +1 = \epsilon''' - 3\phi''' \dots \dots \dots (204f),$$

* The absence of subscripts in some of the Eq. (204a) to (204h) indicates that the same subscript (r) may be suffixed to each symbol.

Twin Chains i.—(3).

$$\alpha'_{r+1} = 2\beta'_r + \alpha'_r = \alpha'''_r, \quad \alpha''_{r+1} = 2\beta'''_r + \alpha'''_r \dots\dots\dots (205a),$$

$$\alpha'_{r+1} = 2C'\alpha'_r - \alpha'_{r-1} = \alpha'''_r, \quad \beta'_{r+1} = 2C'\beta'_r - \beta'_{r-1} = \beta'''_{r+1} \dots\dots\dots (205b),$$

$$\gamma' \sim \delta' = +1 = \gamma''' \sim \delta''' \dots\dots\dots (205c),$$

$$\epsilon' - 2\phi' = +1 = \epsilon''' - 2\phi''' \dots\dots\dots (205d).$$

Single Series ii.—(3).

$$\alpha' \sim \beta' = +1, \quad \alpha'' \sim \beta'' = +3 \dots\dots\dots (206a),$$

$$\beta' = 2\delta'; \quad \beta'' = 2\delta'' \dots\dots\dots (206b),$$

$$\phi' = \gamma' - \alpha' = \epsilon' - \gamma'; \quad \phi'' = \gamma'' - \alpha'' = \epsilon'' - \gamma'' \dots\dots\dots (206c),$$

$$\phi' = \square \text{ or } 2.\square, \quad \phi''_r = 2\square \text{ or } \square \text{ (alterny.)} \dots\dots\dots (206d),$$

$$\alpha'' = 3\alpha' \mp 2, \quad \gamma'' = 3\gamma', \quad \epsilon'' = 3\epsilon' \pm 2 \dots\dots\dots (206e),$$

$$2\phi'_{r+1} = \delta'_{r+1} - \delta'_r, \quad 2\phi''_{r+1} = \delta''_{r+1} - \delta''_r \dots\dots\dots (206f),$$

$$\alpha' = 2\delta' \mp 1, \quad \alpha'' = 2\delta'' \pm 3 \dots\dots\dots (206g).$$

Twin Series ii.—(4).

$$\alpha' \sim \beta' = +1 = \alpha''' \sim \beta'''; \quad \alpha'' \sim \beta'' = +5 = \alpha^{iv} \sim \beta^{iv} \dots\dots\dots (207a),$$

$$\beta' = 2\phi', \quad \beta'' = 2\phi'', \quad \beta''' = 2\phi''', \quad \beta^{iv} = 2\phi^{iv} \dots\dots\dots (207b),$$

$$\delta' = \frac{1}{2}(\gamma' - \alpha') = \frac{1}{3}(\epsilon' - \alpha') = \frac{1}{2}(\gamma''' + \alpha''') = \frac{1}{3}(\epsilon''' + \alpha''') = \delta''' \dots\dots\dots (207c),$$

$$\delta'' = \frac{1}{2}(\gamma'' - \alpha'') = \frac{1}{3}(\epsilon'' - \alpha'') = \frac{1}{2}(\gamma^{iv} + \alpha^{iv}) = \frac{1}{3}(\epsilon^{iv} + \alpha^{iv}) = \delta^{iv} \dots\dots\dots (207d),$$

$$\epsilon' = \gamma' + \delta', \quad \epsilon'' = \gamma'' + \delta'', \quad \epsilon''' = \gamma''' + \delta''', \quad \epsilon^{iv} = \gamma^{iv} + \delta^{iv} \dots\dots\dots (207e).$$

Twin Series ii.—(5).

$$(\alpha', \beta', \gamma', \delta', \epsilon', \phi') = (\alpha''', \beta''', \gamma''', \delta''', \epsilon''', \phi''') \text{ respectively} \dots\dots\dots (208a),$$

$$\gamma' = \gamma''' = 1, \quad \gamma'' = \gamma^{iv} = 3 \dots\dots\dots (208b),$$

$$\epsilon' = 4\phi' + 1 = \epsilon''', \quad \epsilon'' = 4\phi'' - 3, \quad \epsilon^{iv} = 4\phi^{iv} - 3 \dots\dots\dots (208c),$$

$$\phi' = \phi''' = \square \text{ or } 2.\square, \quad \phi'' \text{ \& } \phi^{iv} = 2\square \text{ or } \square \dots\dots\dots (208d),$$

$$\delta'_{r+1} = 16\delta'_r - \delta'_{r-1} = \delta'''_{r+1}, \quad \delta''_{r+1} = 16\delta''_r - \delta''_{r-1} \dots\dots\dots (208e),$$

$$\delta'_r = 2\alpha'_r, \quad \delta''_r = 2\alpha''_r, \quad \delta'''_r = 2\alpha'''_r, \quad \delta^{iv}_r = 2\alpha^{iv}_r, \text{ if } r \text{ is odd} \dots\dots\dots (208f),$$

$$\delta'_r = 2\beta'_r, \quad \delta''_r = 2\beta''_r, \quad \delta'''_r = 2\beta'''_r, \quad \delta^{iv}_r = 2\beta^{iv}_r, \text{ if } r \text{ is even} \dots\dots\dots (208g),$$

$$\phi'_{r+1} = 4\epsilon'_r - \phi'_{r-1}, \quad \phi'_{r+1} = 4\epsilon''_r - \phi''_{r-1}, \quad \phi^{iv}_{r+1} = 4\epsilon^{iv}_r - \phi^{iv}_{r-1} \dots\dots\dots (208h),$$

56. *Even Quartans, Specialty.* When x, y are both odd, then N is always even, so that 2 is here an algebraic factor, and

$$\begin{aligned} N &= 2 \cdot \frac{x^4 + y^4}{2} = \left(\frac{x^4 + y^4}{4} + 1 \right)^2 - \left(\frac{x^4 + y^4}{4} - 1 \right)^2 \\ &= P^2 - Q^2 \dots\dots\dots (209). \end{aligned}$$

Thus every even Quartan has this *special* (algebraic) 2^{ic} partition of type P, Q (wherein P, Q are halves of odd numbers), which yields its own set of four characteristics of Class i: (But there are *no additional* 2^{ic} partitions of the forms (a, b), (c, d), (e, f) due to this special factorisation of N .)

$$Ex. \quad N_0 = 1^4 + 3^4 = 82 = 2.41 = \left(\frac{83}{2}\right)^2 - \left(\frac{81}{2}\right)^2 = \left(\frac{43}{2}\right)^2 - \left(\frac{39}{2}\right)^2.$$

The following Table gives the set of four characteristics (C') and factorisants of Class i., together with the data (x_0, y_0, P_0, Q_0) from which they are derived; prepared as described in Art. 55.*

	x_0, y_0	P_0, Q_0	z'_0, C'	Factorisant	Serial
i.	1, 3	43/2, 39/2	13/2, 41/18	$41x^2 + \frac{1}{2}.1357y^2 = 9z^2$	$x, x; y, y$
	1, 3	-43/2, 39/2	13/2, -5/2	$-5x^2 + \frac{3}{4}y^2 = z^2$	$x, x; z, z$
	3, 1	43/2, 39/2	39/2, 25/2	$(5x)^2 + 69(\frac{3}{2}y)^2 = z^2$	$x, x; y, y$
	3, 1	-43/2, 39/2	39/2, -61/2	$-61x^2 + 413(\frac{3}{2}y)^2 = z^2$	$x, x; z, z$

The following unit-forms may be used with the above factorisants for generating x -Series.

$$\begin{aligned} \left(\frac{109}{13}\right)^2 - 2714 \cdot \left(\frac{2}{13}\right)^2 &= +1; \quad \left(\frac{5}{2}\right)^2 - 2^3 \cdot 1^2 = +1; \\ \left(\frac{13}{10}\right)^2 - 69 \cdot \left(\frac{1}{10}\right)^2 &= +1; \quad \left(\frac{3}{2}\right)^2 - 413 \left(\frac{1}{20}\right)^2 = +1. \end{aligned}$$

Other unit-forms, free from fractions (but leading to higher numbers in x, y), may readily be found.

[The second factorisant above (given by $C' = -5/2$) was obtained also from the Base $N_0 = (1^4 + 1^4)$, see Art. 53, Ex. 2°, and was there shown to yield the present Quartan $N = 1^4 + 3^4$ in its x -Series].

57. Half Quartans, Diophantines. The Diophantine Method of Art. 40 may be also readily applied to Half-Quartans in the Associate form (100*b*)

$$\frac{1}{2}N' = x^4 + 6x^2y^2 + y^4 \dots\dots\dots (210),$$

the (algebraic) 2^{ic} partitions of which are given in (101*a, b*).

Writing, as in Art. 40,

$$\frac{1}{2}N' = (x^2 + Cy^2)^2 + \mu y^2 = (Rx^2 + Sy^2) \dots\dots (210a),$$

and taking $\mu = -1, +1, +2, -2$ (as in Art. 40), the coefficient R of the type factorisant (145) becomes

$$R = (\pm 2C \mp 6), \text{ or } (\mp C \pm 3) \dots\dots\dots (210b).$$

The procedure is then quite similar to that of Art. 40 *et. seq.*, so need not be further developed.

57a. *Allied Quartic Forms. Diophantines.* A similar procedure may be allied to all the quartic forms

$$N = x^4 + kx^2y^2 + y^4 = \Omega \text{ (an odd number)} \dots (211).$$

$$\text{Writing } N = (x^2 + Cy^2)^2 - y^2(Rx^2 + Sy^2) \dots (211a).$$

$$\text{Then, as in Art. 40, } N = F^2 - Q^2$$

$$\text{if } (2C - k)x^2 + (C^2 - 1)y^2 = z^2, \text{ (a square)} \dots (211b).$$

This is the factorisant of Class i. (*cf.* Art. 40). If N has any algebraic 2^{ic} partitions, the factorisants corresponding to such partitions (*i.e.* of Classes ii., iii., &c.) may be formed in a way similar to those of Art. 40.

Ex. An interesting example arises by taking $k = -1$; this gives the Sextan form

$$N = x^4 - x^2y^2 + y^4 = (x^6 + y^6) \div (x^2 + y^2) \dots (211c).$$

It is proposed to develop this Case in a further Paper on the factorisation of Sextans.

58. 2^{ic} *Diophantine connexion with Quartans.* From what precedes—(Art. 40–55)—it is clear that every Quartan gives rise to a whole group of 2^{ic} Diophantines, and that the “4-tan parts” thereof (*i.e.* x, y) are one solution of the whole group, and lead to an infinity of new solutions (x_r, y_r), each of which gives a factorisable Quartan.

Since, in each derived Quartan (N_r), one of the 2^{ic} parts (*i.e.* Q_r, b_r , or a_r, d_r, f_r) $= y_r z_r$; and, since z_r is usually an integer or fraction at same time as C ; it follows that the derived Quartan (N_r) usually has the above 2^{ic} part (Q_r, b_r , or a_r, d_r, f_r) $= y_{r,2}$ or a multiple of y_r , when C is an integer: also since Cy_r^2 should be an integer (Art. 41) it follows that y_r^2 should contain the denominator of C (when C is a fraction).

These conditions limit the nature of the element y_r of all Quartans (N_r) derivable from a given Base (N_0). Hence, although it is certain that the cases are infinitely numerous wherein large composite Quartans are derivable from *prime* Base-Quartans, it seems possible that composite Quartans are *not* necessarily derivable from prime Bases*.

* It is difficult to prove any case of this on account of the number of Diophantines to be solved, and the difficulty of finding the lowest solution of each. *E.g.* The two small Quartans $N_0 = 17, 97$ suffice as Bases to generate *many* of the smaller composite Quartans wherein x & $y < 20$.

59. Inverse Diophantine Problem. The problem now arises—"Given the 4-tan parts (x , y) and two co-factors (L , M) of a composite Quartan (N), to find a Base-Quartan (N_0) thereof (when such exists)."

Proceed as follows:—

Find the whole set of (secondary) characteristics and factorisants associated with the given co-factors (L , M), by the process of Art. 48, 49, 51a. These factorisants may be all included under the type

$$Rx^2 + Sy^2 = z^2,$$

and the general problem is from the known solution (x , y , z) of each, or any, of the factorisants to find a *lower* solution (x' , y' , z'), *i.e.* one leading to a lower Quartan (N'), [so that $N' < N$]. This does not (at present) appear to be *practicable* in the general case (even if possible), especially when R , S are fractions; and the conditions for the existence of a lower solution do not appear to be known.

59a. Serial Case. When the conditions of a Serial solution (160a) are fulfilled, and when the unit-form thereof (161) is also known, the Problem may be completely tested, and solved. Proceed thus—

The formulæ for x_1 , y_1 , z_1 in (162a, b, c) of Art. 45 should be used, *but with the - sign*. If these give values

$$\text{with } \begin{array}{l|l|l} y_1, z_1 < y, z; & \text{or } z_1, x_1 < z, x; & \text{or } x_1, y_1 < x, y; \\ x_1 = x & y_1 = y & z_1 = z \end{array}$$

then x_1 , y_1 , z_1 form a lower solution, leading to $N_1 < N$; otherwise a lower (serial) solution of the factorisant tried is *impossible* (with that particular unit-form)

The same process may be tried again to find a yet lower solution, and so on until the reduction stops; the lowest solution found is a Base-form ($N_0 = x_0^4 + y_0^4$) which will yield the given Quartan (N) by the direct process.

59b. x -Chain Case. When the Given Quartan (say $N_r = x_r^4 + y_r^4 = L_r \cdot M_r$) is a member of an x -chain, the Base-Quartan is easily found. Proceed thus—

$$\text{By (179d)} \quad y_{r-1}y_r = L_r \mp x_r^2, \quad y_r y_{r+1} = M_r \mp x_r^2 \dots \dots \dots (212),$$

$$\text{whence} \quad y_{r-1} = \frac{L_r \mp x_r^2}{y_r}, \quad y_{r+1} = \frac{M_r \mp x_r^2}{y_r}, \quad x_r = x_0 \dots \dots \dots (212a).$$

and the lower solution (x_0 , y_{r-1}) has been found.

$$\text{Again, by (179c)} \quad C' = \frac{y_{r-1} + y_{r+1}}{2y_r} \dots \dots \dots (212b).$$

$$\text{Also, by (179c)} \quad y_{r-2} = 2C' \cdot y_{r-1} - y_r \dots \dots \dots (212c),$$

which gives the next lower values (x_0 , y_{r-2}). By repeated application of this last (204c). y_{r-3} , y_{r-4} , &c., may be found in succession; the lowest value of y so obtained is the Base-value (y_0).

Ex. 1°. [of Art. 59a]. Let $N_r = 1^4 + 112^4 = 3361.46817 = 25089^2 - 21728^2$.

Then $C' = \frac{25089-1}{112.112} = 2, \quad z_r = \frac{21728}{112} = 194.$

The factorisant is $(2x)^2 - 3y^2 = z^2$; the unit-form is $2^2 - 3.1^2 = +1$.

Formulæ (162a, b, c), with the $-$ sign, gives

$$x_{r-1} = x_r = 1; \quad y_{r-1} = 1.194 \sim 2.112 = 30; \quad z_{r-1} = 2.194 \sim 3.1.112 = 52;$$

Thus a *lower* solution (1, 30, 112) has been found.

Repeating this process twice gives (1, 8, 14), then (1, 2, 4), and *no lower* values; so that $N_0 = 1^4 + 2^4 = 17$ is the Base-form—

[See *Ex. 1°* of Art. 52 for the complete x -chain of this].

Ex. 2° [of Art. 59b]. Let $N_0 = 1^4 + 112^4 = 3361.46817$, (as in *Ex. 1°*).

Then $y_{r-1} = \frac{3361-1}{112} = 30; \quad y_{r+1} = \frac{46817-1}{112} = 418; \quad C' = \frac{30+418}{2.112} = 2;$

$$y_{r-2} = 2.2.30 - 112 = 8; \quad y_{r-3} = 2.2.8 - 30 = 2;$$

As y_{r-3} is the *lowest value* obtainable, $y_0 = 2; \quad N_0 = 1^4 + 2^4 = 17;$

Ex. 3°. [General Case]. Let $N = 1921 = 17.113;$

The complete set of 20 (secondary) characteristics is given in the Example at end of Art. 54a. On comparing these with the set of 20 characteristics of $N_0 = 17$ at end of same Article, it will be seen that four of them are alike, viz.

$$N = 17 \text{ has } C'_2 = -\frac{5}{2}; \quad C'_3 = -3, \quad C''_6 = -\frac{5}{2}; \quad C'''_1 = \frac{1}{2};$$

$$N = 1921 \text{ has } C'_2 = -\frac{5}{2}; \quad C'_4 = -3, \quad C''_6 = -\frac{5}{2}; \quad C'''_1 = \frac{1}{2};$$

so that they have four factorisants in common. Thus $N_0 = 17$ is evidently a Base-form of $N_0 = 1921$ in four ways: and would have been found from any one of these four (secondary) factorisants of the latter by the procedure of Art. 59a.

60. *Primary and Secondary Characteristics, Relation.*

The question arises whether any simple (algebraic) relation exists between the primary and secondary characteristics of a composite Quartan. If any such were discoverable, it would lead to a direct factorisation of *given Quartans*.

Example. Taking $N = 5^4 + 6^4 = 1921 = 17.113$. The Table at end of Art. 34a gives the complete sets of 20 primary and 20 secondary characteristics. A variety of simple numerical relations between the primaries and secondaries are easily seen; but, on examining the sets of characteristics for other composite Quartans, the same simple relations are not found, so that this is (at present) an *elusive* Problem.

[The discovery of C for a *given* Quartan (N), i.e. such as to give the expression of N in form $(P^2 - Q^2)$, &c, involves in fact a prior knowledge of P , Q , &c.]

61. Further Chain-properties. Quartan Chains afford—in consequence of the chain properties (Art. 47*b*)—the means of solving the following Problems (already otherwise solved in Art. 30, 31, *q.r.*)—

- i. Forming two equal products of Quartans.
- ii. Forming a square product of Quartans.

If $N_0, N_1, N_2, \dots, N_x$ be a Quartan Chain

wherein N_0 has $L_0 = 1$, and (the last) N_x has* $M_x = N'N''$,

where N', N'' are both Quartans.

Then, by the chain-property—(Art. 47*b*)

$$[N_0 N_2 N_4 \dots N_{2r-2} N_{2r}] = [N_1 N_3 N_5 \dots N_{2r-1}] \cdot M_{2r} \dots (213a),$$

$$[N_1 N_3 N_5 \dots N_{2r-1} N_{2r+1}] = [N_0 N_2 N_4 \dots N_{2r}] \cdot M_{2r+1} \dots (213b).$$

These solve Problem i., since M_{2r} and M_{2r+1} are themselves of form $N'N''$ (by hypothesis).

Again, by the chain-property—

$$[N_0 N_1 N_2 \dots N_r] \cdot M_r = (L_1 L_2 L_3 \dots L_r)^2 \cdot M_r^2 = (M_0 M_1 M_2 \dots M_r)^2 \dots (213c).$$

This solves Problem ii., since M_r is of form $N'N''$ (by hypothesis).

Examples. (See the Table of $N_0 = 17$ at end of Art. 55*a*).

$$1^\circ. \quad N_0 = 17 \text{ with } C' = -\frac{5}{2} \text{ has } M_4 = 17.113.257 = (5^4 + 6^4)(1^4 + 4^4).$$

$$\text{Hence } [(1^4 + 1^4)(1^4 + 14^4)(1^4 + 521^4)] = [(1^4 + 3^4)(1^4 + 67^4)] \cdot (5^4 + 6^4)(1^4 + 4^4).$$

$$2^\circ. \quad N_0 = 17 \text{ with } C' = 5 \text{ has } M_2 = 257.641 = (1^4 + 4^4)(5^4 + 2^4).$$

$$\text{Hence } [(2^4 + 1^4)(2^4 + 129^4)] = [(2^4 + 13^4)] \cdot (1^4 + 4^4)(5^4 + 2^4).$$

3°. The data of Ex. 1° give also

$$\begin{aligned} & [(1^4 + 1^4)(1^4 + 3^4)(1^4 + 14^4)(1^4 + 67^4)(1^4 + 521^4)] \cdot (5^4 + 6^4)(1^4 + 4^4) = \\ & = (2.41.927.2.10733.17.113.257)^2. \end{aligned}$$

The data of Ex. 2° give also

$$[(2^4 + 1^4)(2^4 + 13^4)(2^4 + 129^4)] \cdot (1^4 + 4^4)(5^4 + 2^4) = (17.41.41.257.641)^2.$$

62. Diophantine Factorisation of 8-vans. The Diophantine Methods of Art. 40, *et. seq.*, may be applied to a certain extent to 8-vans, $N = x^8 + y^8$. Writing x^2, y^2 for x, y

* If $M_x = N'$, this would provide simpler solutions; but no instance of $M_x = N'$ has been found (except $M_0 = N'$ by hypothesis).

throughout Art. 40 four Diophantine 4-ic equations, similar to Eq. (141c), (142c), &c., are obtained as the conditions of expressibility of N arithmetically in the four forms

$$N = P^2 - Q^2, \quad a^2 + b^2, \quad c^2 + 2d^2, \quad e^2 - 2f^2 \dots\dots\dots(214).$$

There are the four 4-tic factorisants—

- i. $2C'x^4 + (C'^2 - 1)y^4 = z^2 \dots\dots\dots(214a),$
- ii. $-2C''x^4 - (C''^2 - 1)y^4 = z^2 \dots\dots\dots(214b),$
- iii. $-C'''x^4 - \frac{1}{2}(C'''^2 - 1)y^4 = z^2 \dots\dots\dots(214c),$
- iv. $C''''x^4 + \frac{1}{2}(C''''^2 - 1)y^4 = z^2 \dots\dots\dots(214d).$

Any solution (x, y, z) of these leads—(as in Art. 40a)—to a factorisable 8-van.

63. Suitable Characteristics. As in the case of 4-tans—(in order to use this process at all)—suitable Characteristics must be found, *i.e.* values of C which shall yield solvable factorisants. Writing the 8-van as a Quartan

$$N = x^8 + y^8 = X^4 + Y^4, \quad [X = x^2, \quad Y = y^2] \dots\dots\dots(215),$$

the whole of the procedure, results, and formulæ of Art. 48 are applicable, so far as relates to characteristics (changing of course the x, y thereof into X, Y , or into the new x^2, y^2).

64. Serial Solutions. The only known solutions of much generality are *serial* (Art. 45), *i.e.*, successive solutions depending on a known solution (x_0, y_0, z_0) . The four factorisants may be included in the single type—

$$Rx^4 + Sy^4 = z^2 \dots\dots\dots(216).$$

Then, if x_0, y_0, z_0 be one (known) solution, a second* (x_1, y_1, z_1) is given by

$$x_1 = x_0(3z_0^4 - 4R^2x_0^8), \quad y_1 = y_0(3z_0^4 - 4S^2y_0^8) \dots\dots\dots(217a),$$

$$z_1 = z_0\{4z_0^8 - 3(Rz_0^4 - Sy_0^4)^2\} \dots\dots\dots(217b).$$

Similarly from (x_1, y_1, z_1) , a new solution (x_2, y_2, z_2) may be derived; and so on *ad. inf.* These numbers of course rapidly become enormous.

64a. Simple Serial Solution ($R = k^3$). This involves in the four classes (i., ii., iii., iv.)

- i. $2C' = k^4$; ii. $2C'' = -k^4$; iii. $C''' = -k^4$; iv. $C'''' = k^4 \dots\dots(218),$

* These are Deshoves's formulæ.

* In some cases 2ic factorisants are available for 8-van factorisations. One such case, found by Dr. T. Stuart, is given in *Math. Questions*, &c., from the *Educational Times*, Vol. XIII, 1908, page 96.

whereby the type-factorisant (216) reduces to

$$(kx)^4 + Sy^4 = z^2 \dots\dots\dots(218a).$$

If now (x_0, y_0, z_0) be one solution, then a second (x_1, y_1, z_1) is given by

$$kx_1 = k^4x_0^4 - Sy_0^4, \quad y_1 = 2kx_0y_0z_0, \quad z_1 = z_0^4 + 4Sk^4x_0^4y_0^4 \dots\dots\dots(218b).$$

64*b*. *Simple case of* $y_0 = k\eta_0$. This involves $z_0 = k^2\zeta_0$. It is better then to cancel the factor k^4 throughout at once by writing

$$y = k\eta, \quad z = k^2\zeta \dots\dots\dots(219),$$

whereby the factorisant reduces to the simple form

$$x^4 + S\eta^4 = \zeta^2 \dots\dots\dots(219a),$$

whereof (x_0, η_0, ζ_0) is one solution; and the next is

$$x_1 = x_0^4 - S\eta_0^4; \quad \eta_1 = 2x_0\eta_0\zeta_0; \quad \zeta_1 = \zeta_0^4 + 4Sx_0^4\eta_0^4 \dots\dots\dots(219b),$$

whence $x_1 = x_0^4 - S(y_0/k)^4$; $y_1 = 2x_0y_0z_0/k^2$; $z_1 = (z_0/k^2)^4 + S(x_0y_0/k)^4 \dots\dots(219c)$.

These values of x, y, z will be much lower than the preceding (by the factor k^3). If the resulting x_1, y_1 contain a common factor (k), it may be cancelled, and z_1 must at same time be divided* by k^2 , whereby the yet lower values are obtained—

$$x_1' = x_1/k'; \quad y_1' = y_1/k'; \quad z_1' = z_1/k'^2 \dots\dots\dots(219d).$$

Ex. 1°. Take $N_0 = 1^8 + 1^8 = 2 = (\frac{1}{2})^2 - (\frac{1}{2})^2 = P_0^2 - Q_0^2$.

Here $C' = \frac{1}{2}, \quad R = 1, \quad S = -\frac{3}{4}; \quad k = 1$.

The 4th factorisant is $x^4 - \frac{3}{4}y^4 = z^2$, whereof $x_0 = 1, y_0 = 1, z_0 = \frac{1}{2}$ is the Base-solution.

Hence, by (217), $x_1 : y_1 : z_1 = (1 + \frac{3}{4}) : 2 \cdot \frac{1}{2} : (\frac{1}{16} - 4 \cdot \frac{3}{4}) = 7 : 1 : \frac{4}{16}$;

and, on clearing* fractions, $x_1' : y_1' : z_1' = 7 : 1 : 4$.

whence $P_1 = (7^4 + \frac{1}{2} \cdot 4^4) = 2529, \quad Q_1 = 4^2 \cdot 47 = 752$.

and $N_1 = 7^8 + 4^8 = (2529 - 752)(2529 + 752) = 1777 \cdot (17.193)$.

Again, by (217), $x_2 : y_2 : z_2 = (7^4 + \frac{3}{4} \cdot 4^4) : 2 \cdot 7 \cdot 4 \cdot 47 : (47^4 - 4 \cdot \frac{3}{4} \cdot 7^4 \cdot 4^4)$
 $= 2593 : 2632 : 3035713$.

Hence $P_2 = 2593^4 + \frac{1}{2} \cdot 2632^4, \quad Q_2 = 2632^2 \cdot 3035713$.

And $N_2 = 2593^8 + \frac{1}{2} \cdot 2632^8 = (P_2 - Q_2)(P_2 + Q_2)$.

Ex. 2°. Take $N_0 = 1^8 + 2^8 = 257 = 129^2 - 128^2$.

* In clearing fractions, if x_1, y_1 be multiplied by μ , z_1 must be multiplied by μ^2 .

Here $C'=8$, $R=16$, $k=2$, whence the 4^{ic} factorisant

$$(2x)^4 + 63y^4 = z^2, \text{ with } x_0=1, y_0=2, z_0=32.$$

Here $y_0=2$ is divisible by $k=2$; so this falls under Art. 64b.

Writing $y=2\eta$, $z=4\zeta$, the reduced factorisant is

$$x^4 + 63\eta^4 = \zeta^2, \text{ with } x_0=1, \eta_0=1, \zeta_0=8, \text{ as Base-solution.}$$

Hence by (219b) $x_1=62$, $\eta_1=16$, $\zeta_1=4348$, $y_1=32$, $z_1=17392$.

The factor 2 may now be cancelled from x_1, y_1 , and 4 from z_1 giving the *reduced solution* $x_1'=31$, $y_1'=16$, $z_1'=4348$.

$$\text{Here } P_1' = 31^4 + 8.16^4 = 1447809; \quad Q_1' = y_1'^2 z_1' = 143088;$$

Hence

$$N_1 = 31^8 + 16^8 = (P_1' - Q_1')(P_1' + Q_1') = 334721 \cdot 256089 = 334721 \cdot (17.97.1553).$$

$$\text{Again, by (217); } x_2 : \frac{1}{2}y_2 = \{31^4 - 63(\frac{16}{2})^4\} : 2.31. \frac{16}{2}, \frac{4.3.4.8}{2};$$

$$\text{whence } x_2 : y_2 = 665473 : 1078304.$$

It is evident that though this Method is of the same generality as applied to 8-vans, as in the previous application to 4-tans, still the numbers usually run too high for the Method to be of much practical use for 8-vans.

65. High Primes (Tab. IX). Here follows a List of High Primes ($>9.10^6$) which are either Quartans, Half-Quartans, or Factors thereof, *i.e.*

$$p = (x^4 + y^4), \text{ or } \frac{1}{2}(x^4 + y^4), \text{ or } \frac{1}{\mu}(x^4 + y^4),$$

detected in course of preparing this Paper (in addition to those already reported, in Art. 23, Tab. III.—VII. of the first Paper).

p	x	y	μ	p	x	y	μ
10418833	*($2^{34} + 6.2^{17} + 1$) ;			100060001	99,	101 ;	2
10562737	999,	1001 ;	2μ	10845953	1,	127^2 ;	μ
16548577	1,	18957 ;	2μ	123993929	1,	958 ;	μ
16801793	63,	65 ;	2				
49195721	1,	16800 ;	μ	1142991841	1,	815 ;	μ
63172849	*($2^{30} + 6.2^{15} + 1$) ;			1150215097	1,	754 ;	μ
65028097	8191,	16 ;	μ	1269398609	1,	46^2 ;	μ
91422241	3039,	2 ;	μ	1505882353	1,	20^4 ;	μ

* These are Quasi Half-Quartans, Art. 36.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

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XXIV.

Oscillating cases of Dirichlet's Integral (continued).

§ 1. IN Note XXIII. I considered the integral

$$(1) \quad J(\lambda) = \int_0^{\xi} F(x) \phi(1/x) \frac{\cos \psi(1/x) \sin \lambda x}{x} dx,$$

where ϕ and ψ are members of the logarithmic scale and $F(x)$ regular in every way for $x=0^*$. I confined myself to the case in which $\psi(1/x)$ bends to ∞ with $1/x$ not more rapidly than $\log(1/x)$. This limitation is essential for the success of the method which I used, and the results are, as I pointed out, quite different in the contrary case. But even this case I did not discuss completely, as my analysis applied only to the cases in which $\phi(1/x) \equiv 1$, when $J(\lambda)$ oscillates finitely, or $\phi(1/x) \rightarrow 0$, when $J(\lambda) \rightarrow 0$.

In this note I propose to complete my results by discussing the case in which

$$\phi(1/x) \rightarrow \infty$$

as $x \rightarrow 0$. The method of my former note can still be used with certain modifications.†

§ 2. Let us suppose, in the first instance, that $F(x) \equiv 1$ and

$$\phi(u) = u^a (l_1 u)^{a_1} (l_2 u)^{a_2} \dots (l_k u)^{a_k},$$

* It is of course supposed that ξ is small enough to ensure that the logarithmic factors are continuous save for $x=0$.

† At the bottom of p. 5 of my former note I stated that "considerably more elaborate methods are needed" to discuss this case. This seems, in the light of my subsequent investigations, to be an over-statement.

I take this opportunity of making some corrections in my former note. On p. 4, at the end of § 4, for $\lambda/\mu < \rho < \lambda\mu$ read $\lambda/\mu < 2\rho < \lambda\mu$. The same correction should be made at the beginning of § 6 (p. 5, l. 1); and in consequence $\frac{\log \mu}{\chi(\lambda/\mu)}$, which occurs twice immediately below, should be replaced by

$$\frac{\log(2\mu)}{\chi(\lambda/2\mu)}.$$

These alterations in no way affect the argument.

There are also obvious misprints on p. 3, l. 3, where $(1 - \cos \lambda x)/x$ should be replaced by $(1 - \cos \lambda x) \lambda$; and on p. 4, first footnote, l. 1, where "as $\lambda \rightarrow \infty$ " should be read for "of $\lambda \rightarrow \infty$."

where $l_1 u \equiv \log u$, $l_2 u \equiv l_1 l_1 u$, ... I shall suppose, in what follows, that $\alpha_3 = \alpha_4 = \dots = \alpha_k = 0$, so that

$$\phi(u) = u^a (\log u)^{a_1} (\log \log u)^{a_2}.$$

This hypothesis enables us to avoid a good deal of unessential complexity of notation, and there is no additional difficulty of principle involved in the treatment of the general case.

If $a > 1$ the integral (1) is not convergent, while if $a < 1$ it is convergent for all values of α_1 and α_2 . If $a = 1$ its convergence or divergence depends upon the values of α_1 and α_2 and the precise form of $\psi(1/x)$. We shall suppose $a < 1^*$. In this case, in the notation of Note XXIII., $\xi f(\xi) \rightarrow 0$ as $\xi \rightarrow 0$, and we have, as in § 3 of that note, to discuss the integral

$$(2) \quad I(\lambda) = \int_0^\epsilon \theta'(u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx,$$

which we again divide into the three parts $I_1(\lambda)$, $I_2(\lambda)$, and $I_3(\lambda)$. Here

$$\theta(u) = u f(x) = u^{a+1} (\log u)^{a_1} (\log \log u)^{a_2} \frac{\cos}{\sin} \psi(u).$$

As in § 5 of Note XXIII. I shall work with the sine rather than the cosine. Then it is easy to see that

$$(3) \quad \theta'(u) = (a+1) u^a (\log u)^{a_1} (\log \log u)^{a_2} \sin \psi(u) (1 + \epsilon_u),$$

where $\epsilon_u \rightarrow 0$ as $u \rightarrow \infty$, and in fact $|\epsilon_u|$ is less than the lesser of the functions

$$K/\log u, \quad K/\chi(u),$$

where $\chi(u)$ is defined as in § 6 of Note XXIII., viz. by the equation $\psi'(u) = 1/\{\chi(u)\}$.

In $I_2(\lambda)$ the subject of integration is the product of $\sin \psi(u)$ by an essentially positive function. Hence

$$\begin{aligned} I_2(\lambda) &= (a+1) (\log \rho)^{a_1} (\log \log \rho)^{a_2} \sin \psi(\rho) (1 + \epsilon_\rho) \\ &\quad \times \frac{1}{2} \lambda \int_{\delta_1}^{\delta_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 x^{-a} dx \\ &= (a+1) 2^{-a} \lambda^a (\log \rho)^{a_1} (\log \log \rho)^{a_2} \sin \psi(\rho) (1 + \epsilon_\rho) \\ &\quad \times \int_{1/\mu}^\mu \left(\frac{\sin t}{t} \right)^2 t^{-a} dt. \end{aligned}$$

* For a discussion of the convergence or divergence of infinite integrals and series involving slowly oscillating factors, the reader may be referred to a paper by Mr. Bromwich, *Proc. L.M.S.*, N.S., vol. 6, p. 327.

As in § 6 of Note XXIII. we can show that, if $\mu \rightarrow \infty$ sufficiently slowly,

$$\sin \psi(\rho) = \sin \psi(\lambda) + \epsilon_\lambda.$$

And if $\mu \rightarrow \infty$ more slowly than any power of λ , so that $(\log \mu)/(\log \lambda) \rightarrow 0$, we have $(\log \rho)^{a_1} = (\log \lambda + \log \nu)^{a_1}$, where $|\log \nu| < \log 2\mu$, or

$$(\log \rho)^{a_1} = (\log \lambda)^{a_1} (1 + \epsilon_\lambda),$$

$$\begin{aligned} \text{and } (\log \log \rho)^{a_2} &= \left\{ \log \log \lambda + \log \left(1 + \frac{\log \nu}{\log \lambda} \right) \right\}^{a_2} \\ &= (\log \log \lambda)^{a_2} (1 + \epsilon_\lambda). \end{aligned}$$

Finally

$$\int_{1/\mu}^{\mu} \left(\frac{\sin t}{t} \right)^2 t^{-a} dt \rightarrow \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 t^{-a} dt = 2^a \sin \frac{1}{2} a \pi \Gamma(-1-a),$$

if $a > 0$, or $\frac{1}{2}\pi$ if $a = 0$. Hence we arrive at the formula

$$\begin{aligned} (4) \quad I_2(\lambda) &= -\sin \frac{1}{2} a \pi \Gamma(-a) \lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \\ &\quad \times (1 + \epsilon_\lambda) \{ \sin \psi(\lambda) + \epsilon_\lambda \}. \end{aligned}$$

§ 3. We have now to consider the integrals $I_1(\lambda)$ and $I_2(\lambda)$, defined by the equations

$$I_1(\lambda) = \int_0^{\delta_1} \theta'(u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx, \quad I_2(\lambda) = \int_{\delta_2}^{\xi} \theta'(u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx.$$

In virtue of the equation (3), we have

$$|\theta'(u)| < K u^a (\log u)^{a_1} (\log \log u)^{a_2},$$

and so

$$\begin{aligned} |I_1(\lambda)| &< K \int_0^{\delta_1} u^a (\log u)^{a_1} (\log \log u)^{a_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx \\ &< K \lambda^a \int_0^{1/\mu} t^{-a} \left(\log \frac{\lambda}{2t} \right)^{a_1} \left(\log \log \frac{\lambda}{2t} \right)^{a_2} \left(\frac{\sin t}{t} \right)^2 dt. \end{aligned}$$

But throughout the range of integration $\lambda/2t$ is large, and

$$\begin{aligned} \left(\log \frac{\lambda}{2t} \right)^{a_1} &< K (\log \lambda)^{a_1} \left(\log \frac{1}{t} \right)^{a_1}, \\ \left(\log \log \frac{\lambda}{2t} \right)^{a_2} &< K \left(\log \log \lambda + \log \log \frac{1}{t} \right)^{a_2} \\ &< K (\log \log \lambda)^{a_2} \left(\log \log \frac{1}{t} \right)^{a_2}, \end{aligned}$$

so that

$$|I_1(\lambda)| < K\lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \\ \times \int_0^{\delta_1} t^{-a} \left(\log \frac{1}{t}\right)^{a_1} \left(\log \log \frac{1}{t}\right)^{a_2} \left(\frac{\sin t}{t}\right)^2 dt.$$

$$\text{Hence } I_1(\lambda) / \{\lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2}\} \rightarrow 0$$

as $\lambda \rightarrow \infty$; and in the same way we can prove that

$$I_3(\lambda) / \{\lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2}\} \rightarrow 0.$$

Thus finally we may write

$$(5) \quad I(\lambda) = -\sin \frac{1}{2} a \pi \Gamma(-a) \lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \\ \times \{\sin \psi(\lambda) + \epsilon_\lambda\},$$

a formula which exhibits the nature of the oscillations of $I(\lambda)$ as $\lambda \rightarrow \infty$. The same formula holds for $J(\lambda)$.

§ 4. So far we have supposed $F(x) \equiv 1$. The general case may be at once reduced to dependence on this special case if (as it is sufficient for our present purposes to suppose) $F(x)$ has a continuous derivative near $x=0$. We have only to express $F(x)$ in the form

$$F(0) + xF'(\theta x) \quad (0 < \theta < 1),$$

and to consider the two terms separately. Our final result is that the asymptotic formula for $J(\lambda)$, when $\psi(1/x)$ is lower in the scale than $\log(1/x)$, and

$$\phi(1/x) = x^{-a} \{l_1(1/x)\}^{a_1} \{l_2(1/x)\}^{a_2} \dots \{l_k(1/x)\}^{a_k},$$

where $0 < a < 1$, is

$$(6) \quad -\sin \frac{1}{2} a \pi \Gamma(-a) F(0) \lambda^a (l_1 \lambda)^{a_1} (l_2 \lambda)^{a_2} \dots (l_k \lambda)^{a_k} \\ \times \left\{ \frac{\cos}{\sin} \psi(\lambda) + \epsilon_\lambda \right\};$$

a result which completes that stated at the end of § 7 of Note XXIII. This result, so far as I am aware, was not given by Du Bois Reymond.

§ 5. I shall now illustrate these general results by means of some particular integrals.

Let

$$(7) \quad f_{r,s}(t) = \Gamma(s) \sum_0^\infty \frac{t^r}{(\nu+r)^s \nu!} = \int_0^1 e^{tx} x^{r-1} \left\{ \log \left(\frac{1}{x} \right) \right\}^{s-1} dx,$$

where $R(r) > 0$ and $R(s) > 0$. In a paper published in 1905 in the *Proceedings of the London Mathematical Society** I investigated the behaviour of this function for large values of t , real or complex. This problem has also been considered independently by Dr. Barnes†.

In what follows I shall be concerned exclusively with large and purely imaginary values of t . In the paper already quoted I proved that for such values of t ,

$$(8) \quad f_{r,s}(t) = \Gamma(r) (-t)^{-r} \{\log(-t)\}^{s-1} (1 + \epsilon_t) + \Gamma(s) t^{-s} e^t (1 + \epsilon'_t),$$

where ϵ_t and ϵ'_t tend to zero as $|t| \rightarrow \infty$. It will be observed that the dominant term is the first or second according as $R(s) > R(r)$ or $R(s) < R(r)$. In this formula

$$(-t)^{-r} = \exp\{-r \log(-t)\} = \exp[-r\{\log|t| - \tfrac{1}{2}\epsilon\pi i\}],$$

$$t^{-s} = \exp(-s \log t) = \exp[-s\{\log|t| + \tfrac{1}{2}\epsilon\pi i\}],$$

where $\epsilon = +1$ or $\epsilon = -1$ according as $t/i > 0$, or $t/i < 0$.

If in (8) we put $t = i\lambda$ and $t = -i\lambda$, where $\lambda > 0$, subtract and divide by $2i$, we obtain

$$\begin{aligned} (9) \quad \int_0^1 x^r \left\{ \log \left(\frac{1}{x} \right) \right\}^{s-1} \frac{\sin \lambda x}{x} dx &= \frac{1}{2i} \{f_{r,s}(i\lambda) - f_{r,s}(-i\lambda)\} \\ &= \Gamma(r) \sin(\tfrac{1}{2}r\pi) \lambda^{-r} (\log \lambda)^{s-1} (1 + \epsilon_\lambda) \\ &\quad + \Gamma(s) \lambda^{-s} \sin(\lambda - \tfrac{1}{2}s\pi) (1 + \epsilon_\lambda). \end{aligned}$$

This formula has been established on the assumption that $R(r) > 0$ and $R(s) > 0$; but it holds for $R(r) > -1$, as may be proved by a direct investigation similar to that by which the formula (8) was established.

§ 6. Since $R(s) > 0$, the second term on the right-hand side of (9) tends to zero as $\lambda \rightarrow \infty$. If $R(r) > 0$, or $R(r) = 0$, $0 < R(s) < 1$, the first term also tends to zero. But if $R(r) < 0$, or $R(r) = 0$, $R(s) \geq 1$, we have the asymptotic formula

$$(10) \quad \int_0^1 x^r \left\{ \log \left(\frac{1}{x} \right) \right\}^{s-1} \frac{\sin \lambda x}{x} dx \sim \Gamma(r) \sin(\tfrac{1}{2}r\pi) \lambda^{-r} (\log \lambda)^{s-1}.$$

* *Proc. L.M.S.*, N.S., vol. ii, pp. 401 *et seq.*

† *Phil. Trans. (A)*, vol. 206, pp. 249–297.

Suppose in particular that r is real, and let $s = \rho + i\sigma$. Then if $r < 0$ or $r = 0$, $\rho \geq 1$, we obtain the formula

$$(11) \quad \int_0^1 x^r \left\{ \log \left(\frac{1}{x} \right) \right\}^{\rho-1} \frac{\cos \left\{ \sigma \log \log \left(\frac{1}{x} \right) \right\} \sin \lambda x}{\sin} dx \\ \sim \Gamma(r) \sin \left(\frac{1}{2} r \pi \right) \lambda^{-r} (\log \lambda)^{\rho-1} \frac{\cos}{\sin} (\sigma \log \log \lambda).$$

In particular, if $r = 0$, $\rho = 1$, we obtain

$$(12) \quad \int_0^1 \frac{\cos \left\{ \sigma \log \log \left(\frac{1}{x} \right) \right\} \sin \lambda x}{\sin} dx \sim \frac{1}{2} \pi \frac{\cos}{\sin} (\sigma \log \log \lambda)$$

in agreement with equation (6) of Note XXIII. The more general result (11) is obviously in agreement with the general formula (5) of § 3*.

§ 7. It remains to consider the case in which

$$\psi(1/x) = m \log(1/x).$$

In this case

$$\theta(u) = u f(x) = u^{a+1} (\log u)^{a_1} (\log \log u)^{a_2} \sin(m \log u)$$

and

$$\theta'(u) = u^a (\log u)^{a_1} (\log \log u)^{a_2} \{ (a+1) \sin(m \log u) \\ + m \cos(m \log u) \} (1 + \varepsilon_u) \\ = u^a (\log u)^{a_1} (\log \log u)^{a_2} (h u^{mi} + k u^{-mi}) (1 + \varepsilon_u),$$

where $h = \frac{1}{2} \{ m - (a+1)i \}$, $k = \frac{1}{2} \{ m + (a+1)i \}$.

Thus if

$$L(u) = (\log u)^{a_1} (\log \log u)^{a_2}, \quad M(u) = h u^{mi} + k u^{-mi},$$

we have

$$I_2(\lambda) = \int_{\delta_1}^{\delta_2} u^a L(u) M(u) (1 + \varepsilon_u) \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx.$$

This equation may be written in the form

$$I_2(\lambda) = L(\lambda) (1 + \varepsilon_\lambda) \int_{\delta_1}^{\delta_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 u^a M(u) \frac{1}{2} \lambda dx.$$

* The symbol of asymptotic equality is not used quite strictly here. For example

$$F(\lambda) = \sin \{ \psi(\lambda) \} + \varepsilon_\lambda$$

does not involve $F(\lambda) \sim \sin \{ \psi(\lambda) \}$ strictly, since for certain values of λ the sine will be the smaller of the two terms. But the meaning of the symbol, as here used, will be clear enough.

It should however be observed that this step is no longer a mere corollary from the first Mean Value Theorem, inasmuch as $M(u)$ is not of invariable sign. But, by the second Mean Value Theorem,

$$I_1(\lambda) = L(\delta_1) (1 + \epsilon_\lambda) \int_{\delta_1}^{\xi} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 u^a M(u) \frac{1}{2} \lambda dx \\ + L(\delta_2) (1 + \epsilon_\lambda) \int_{\xi}^{\delta_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 u^a M(u) \frac{1}{2} \lambda dx,$$

where $\delta_1 < \xi < \delta_2$; and the result follows if we observe that $L(\delta_1)$ and $L(\delta_2)$ can each be expressed in the form

$$L(\lambda) (1 + \epsilon_\lambda).$$

Now
$$\int_{\delta_1}^{\delta_2} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 u^a M(u) \frac{1}{2} \lambda dx = H + K,$$

where

$$H = h \int_{\delta_1}^{\delta_2} u^{a+mi} \left(\frac{\sin \frac{1}{2} \lambda x}{\frac{1}{2} \lambda x} \right)^2 \frac{1}{2} \lambda dx \\ = h \left(\frac{1}{2} \lambda \right)^{a+mi} (1 + \epsilon_\lambda) \int_0^\infty t^{-a-mi} \left(\frac{\sin t}{t} \right)^2 dt \\ = h \lambda^{a+mi} \sin \left\{ \frac{1}{2} (a + mi) \pi \right\} \Gamma(-1 - a - mi) (1 + \epsilon_\lambda),$$

and similarly

$$K = k \lambda^{a-mi} \sin \left\{ \frac{1}{2} (a - mi) \pi \right\} \Gamma(-1 - a + mi) (1 + \epsilon_\lambda).$$

Hence we obtain, as an asymptotic expression of $I_1(\lambda)$, the function

$$\lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \left[h \lambda^{mi} \sin \left\{ \frac{1}{2} (a + mi) \pi \right\} \Gamma(-1 - a - mi) \right. \\ \left. + k \lambda^{-mi} \sin \left\{ \frac{1}{2} (a - mi) \pi \right\} \Gamma(-1 - a + mi) \right] \\ = \lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \left[\frac{1}{2} i \lambda^{mi} \sin \left\{ \frac{1}{2} (a + mi) \pi \right\} \Gamma(-a - mi) \right. \\ \left. - \frac{1}{2} i \lambda^{-mi} \sin \left\{ \frac{1}{2} (a - mi) \pi \right\} \Gamma(-a + mi) \right].$$

We can show, as before, that this function also gives an asymptotic representation of $J(\lambda)$.

Finally, if we put

$$(13) \quad \sin \left\{ \frac{1}{2} (a + mi) \pi \right\} \Gamma(-a - mi) = P + Qi,$$

we obtain

$$(14) \quad -\lambda^a (\log \lambda)^{a_1} (\log \log \lambda)^{a_2} \{ P \sin(m \log \lambda) + Q \cos(m \log \lambda) \}.$$

§ 8. Now let us, in the equation (10) of § 6, write

$$r = -a - mi,$$

and suppose s real. Then we obtain

$$\begin{aligned} \int_0^1 \left(\frac{1}{x}\right)^a \left\{ \log \left(\frac{1}{x}\right) \right\}^{s-1} \exp \left\{ mi \log \left(\frac{1}{x}\right) \right\} \frac{\sin \lambda x}{x} dx \\ \sim -\Gamma(-a-mi) \sin \left\{ \frac{1}{2} (a+mi) \pi \right\} \lambda^{a+mi} (\log \lambda)^{s-1} \\ = -(P+Qi) \lambda^a (\log \lambda)^{s-1} \exp(mi \log \lambda). \end{aligned}$$

Thus

$$\begin{aligned} (15) \quad \int_0^1 \left(\frac{1}{x}\right)^a \left\{ \log \left(\frac{1}{x}\right) \right\}^{s-1} \cos \left\{ m \log \left(\frac{1}{x}\right) \right\} \frac{\sin \lambda x}{x} dx \\ \sim \{-P \cos(m \log \lambda) + Q \sin(m \log \lambda)\} \lambda^a (\log \lambda)^{s-1}. \end{aligned}$$

$$\begin{aligned} (16) \quad \int_0^1 \left(\frac{1}{x}\right)^a \left\{ \log \left(\frac{1}{x}\right) \right\}^{s-1} \sin \left\{ m \log \left(\frac{1}{x}\right) \right\} \frac{\sin \lambda x}{x} dx \\ \sim \{-P \sin(m \log \lambda) - Q \cos(m \log \lambda)\} \lambda^a (\log \lambda)^{s-1}. \end{aligned}$$

In particular, if $a=0$, $s=1$, we obtain

$$\begin{aligned} (17) \quad \int_0^1 \cos \left\{ m \log \left(\frac{1}{x}\right) \right\} \frac{\sin \lambda x}{x} dx \\ \sim -P \cos(m \log \lambda) + Q \sin(m \log \lambda). \end{aligned}$$

$$\begin{aligned} (18) \quad \int_0^1 \sin \left\{ m \log \left(\frac{1}{x}\right) \right\} \frac{\sin \lambda x}{x} dx \\ \sim -P \sin(m \log \lambda) - Q \cos(m \log \lambda), \end{aligned}$$

where $P+Qi = \sin(\frac{1}{2}m\pi i) \Gamma(-mi)$;

so that, if $\Gamma(mi) = \lambda + \mu i$, we have

$$\begin{aligned} P+Qi &= i \sinh(\tfrac{1}{2}m\pi) (\lambda - \mu i), \\ P &= \mu \sinh \tfrac{1}{2}m\pi, \quad Q = \lambda \sinh \tfrac{1}{2}m\pi. \end{aligned}$$

These equations agree with those found otherwise in Note XXIII. (p. 7).

We can, of course, by supposing both r and s to be complex, obtain more general formula in which the oscillatory part of the subject of integration is of the type

$$\frac{\cos}{\sin} \left\{ m \log \left(\frac{1}{x}\right) + n \log \log \left(\frac{1}{x}\right) \right\}.$$

§ 9. Further simple and interesting examples of oscillating Dirichlet's integrals may be obtained by the help of the series which I investigated in a paper "On certain oscillating series" in vol. XXXVIII. of the *Quarterly Journal*.* I proved

* Q.J., l.c., p. 269.

there for example, that the series

$$F_{\alpha}(x) = e^{-x} - e^{-\alpha x} + e^{-\alpha^2 x} - \dots,$$

where $\alpha > 1$, oscillates finitely as $x \rightarrow 0$, the series being in fact equal to

$$\sum_0^{\infty} \frac{(-)^n}{n!} \frac{x^n}{1+\alpha^n} + \frac{1}{\log \alpha} \sum_{-\infty}^{\infty} \Gamma \left\{ -\frac{(2k+1)\pi i}{\log \alpha} \right\} \exp \left\{ \frac{(2k+1)\pi i}{\log \alpha} \log x \right\}.$$

It is easy to show that

$$\int_0^{\infty} F_{\alpha}(x) \frac{\sin \lambda x}{x} dx = \arctan \lambda - \arctan \left(\frac{\lambda}{\alpha} \right) + \arctan \left(\frac{\lambda}{\alpha^2} \right) - \dots,$$

and that the latter series oscillates finitely as $\lambda \rightarrow \infty$. In fact, if $\lambda = \alpha^n$, we have

$$\sum_0^{\infty} (-)^{\nu} \arctan (\alpha^{n-\nu}) = \left(\sum_0^{2n} + \sum_{2n+1}^{\infty} \right) (-)^{\nu} \arctan (\alpha^{n-\nu}) = S_1 + S_2,$$

say. But

$$\begin{aligned} S_1 &= \arctan \alpha^n - \arctan \alpha^{n-1} + \dots + (-1)^{n-1} \arctan \alpha \\ &\quad + \arctan \alpha^{-n} - \arctan \alpha^{-(n-1)} + \dots + (-1)^{n-1} \arctan \alpha^{-1} \\ &\quad + (-1)^n \frac{1}{4} \pi \\ &= \frac{1}{2} \pi - \frac{1}{2} \pi + \dots + (-1)^{n-1} \frac{1}{2} \pi + (-1)^n \frac{1}{4} \pi = \frac{1}{4} \pi. \end{aligned}$$

And $S_2 = -\arctan \alpha^{-n-1} + \arctan \alpha^{-n-2} - \dots$,

so that $-\arctan \alpha^{-n-1} < S_2 < 0$,

and $S_2 \rightarrow 0$. Thus the series tends to $\frac{1}{4} \pi$ if $\lambda \rightarrow \infty$ through a series of values of the type α^n . On the other hand, if $\lambda = \alpha^{n+\frac{1}{2}}$, we obtain

$$\begin{aligned} \sum_0^{\infty} (-)^{\nu} \arctan (\alpha^{n+\frac{1}{2}-\nu}) &= \left(\sum_0^{2n+1} + \sum_{2n+2}^{\infty} \right) (-)^{\nu} \arctan (\alpha^{n+\frac{1}{2}-\nu}) \\ &= S_1 + S_2, \end{aligned}$$

say. As before, we see that $S_2 \rightarrow 0$. Also

$$\begin{aligned} S_1 &= \arctan \alpha^{n+\frac{1}{2}} - \arctan \alpha^{n-\frac{1}{2}} + \dots + (-1)^n \arctan \alpha^{\frac{1}{2}} \\ &\quad - \arctan \alpha^{-n-\frac{1}{2}} + \arctan \alpha^{-n+\frac{1}{2}} + \dots - (-1)^n \arctan \alpha^{-\frac{1}{2}} \\ &= \frac{1}{2} \pi - \frac{1}{2} \pi + \dots + (-1)^n \frac{1}{2} \pi \\ &\quad + 2(-1)^{n+1} \{ \arctan \alpha^{-\frac{1}{2}} - \arctan \alpha^{-\frac{3}{2}} + \dots + (-1)^n \arctan \alpha^{-n-\frac{1}{2}} \} \\ &= 2\sigma \text{ or } \frac{1}{2} \pi - 2\sigma, \end{aligned}$$

according as n is odd or even, σ being defined by the equation

$$\sigma = \arctan \alpha^{-\frac{1}{2}} - \arctan \alpha^{-\frac{3}{2}} + \dots + (-1)^n \arctan \alpha^{-n-\frac{1}{2}}.$$

Evidently $0 < \sigma < \arctan \alpha^{-\frac{1}{2}},$

and if $\arctan \alpha^{-\frac{1}{2}} < \frac{1}{8}\pi,$

or $\alpha > \cot^2 \frac{1}{8}\pi = \frac{\sqrt{2}+1}{\sqrt{2}-1},$

the limits of oscillation of the series are at any rate wider than

$$2 \arctan \alpha^{-\frac{1}{2}}, \quad \frac{1}{2}\pi - 2 \arctan \alpha^{-\frac{1}{2}}.$$

If α does not satisfy this condition a somewhat less elementary method of proof, such as that used in § 8 of my paper in the *Quarterly Journal* already quoted, appears to be necessary.

THE JACOBIAN OF FOUR QUADRICS.

By C. M. Jessop.

IF $S_1=0, S_2=0, S_3=0, S_4=0$ are any four quadrics, it has been shown by Reye* that the system of quadrics

$$F \equiv \lambda S_1 + \mu S_2 + \nu S_3 + \rho S_4 = 0$$

becomes a pair of planes for 10 sets of values of the quantities $\lambda, \mu, \nu, \rho.$ It is clear that the Jacobian, $\frac{d(S_1 S_2 S_3 S_4)}{d(x_1 x_2 x_3 x_4)} = 0,$ of

the system will contain the line of intersection of each of these pairs of planes; hence the Jacobian contains 10 lines.

The following proof may also be given of the existence of 10 lines on the Jacobian of any four quadrics:—

If $F=0$ represents two planes, three independent relationships will hold among its coefficients giving a finite number of sets of values of the ratios $\lambda:\mu:\nu:\rho.$ Assuming that there are at least two, which may be done, since none of the relationships are linear, take two such pairs of planes to form the tetrahedron of reference; then the given system of quadrics may be written in the form

$$\lambda S_1 + \mu S_2 + \nu x_1 x_2 + \rho x_3 x_4 = 0.$$

* See Reye, *Crelle's Journal*, vol. 86 (1879).

The Jacobian of the system is then

$$\begin{vmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_4 & x_3 \\ \frac{\partial S_1}{\partial x_1} & \frac{\partial S_1}{\partial x_2} & \frac{\partial S_1}{\partial x_3} & \frac{\partial S_1}{\partial x_4} \\ \frac{\partial S_2}{\partial x_1} & \frac{\partial S_2}{\partial x_2} & \frac{\partial S_2}{\partial x_3} & \frac{\partial S_2}{\partial x_4} \end{vmatrix} = 0.$$

This may be written in the form

$$\begin{aligned} & \left(x_1 \frac{\partial S_1}{\partial x_1} - x_2 \frac{\partial S_1}{\partial x_2} \right) \left(x_3 \frac{\partial S_2}{\partial x_3} - x_4 \frac{\partial S_2}{\partial x_4} \right) \\ & - \left(x_1 \frac{\partial S_2}{\partial x_1} - x_2 \frac{\partial S_2}{\partial x_2} \right) \left(x_3 \frac{\partial S_1}{\partial x_3} - x_4 \frac{\partial S_1}{\partial x_4} \right) = 0. \end{aligned}$$

Now we may regard the coefficients of the quadric S_1 as containing one arbitrary constant linearly; let us consider the condition that the quadrics

$$u \equiv x_1 \frac{\partial S_1}{\partial x_1} - x_2 \frac{\partial S_1}{\partial x_2} = 0, \quad v \equiv x_3 \frac{\partial S_1}{\partial x_3} - x_4 \frac{\partial S_1}{\partial x_4} = 0,$$

should have one generator in common. The condition required is that the discriminant of $\kappa u + v$ should be a *perfect square*. If we suppose $S_1 \equiv \Sigma a_{ik} x_i x_k$, this discriminant is

$$\begin{vmatrix} 2a_{11}\kappa & 0 & (\kappa+1)a_{13} & (\kappa-1)a_{14} \\ 0 & -2a_{22}\kappa & (1-\kappa)a_{23} & -(\kappa+1)a_{24} \\ (\kappa+1)a_{13} & (1-\kappa)a_{23} & 2a_{33} & 0 \\ (\kappa-1)a_{14} & -(\kappa+1)a_{24} & 0 & -2a_{44} \end{vmatrix},$$

which, when expanded, assumes the form

$$\Delta^2 \kappa^4 + A \kappa^3 + B \kappa^2 + A \kappa + \Delta^2,$$

where
$$\Delta = \begin{vmatrix} a_{13} a_{14} \\ a_{23} a_{24} \end{vmatrix}.$$

This expression will be a perfect square if

$$B \Delta^2 = A^2 + 2 \Delta^4.$$

This gives an equation of degree 8 in the quantities a_{ik} , i.e., of degree 8 in the arbitrary constant which the a_{ik} involve linearly.

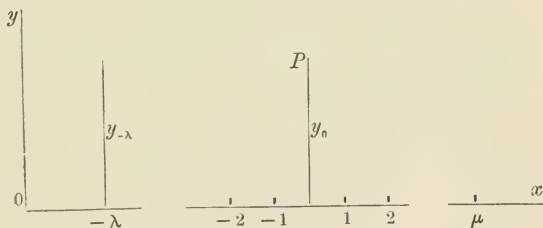
Thus we get 8 pairs of quadrics, each pair having a common generator, which, therefore, also belong to the Jacobian; counting the original two lines $x_1=0$, $x_2=0$; $x_3=0$, $x_4=0$, it is seen that the Jacobian possesses 10 lines.

A FORMULA IN INTERPOLATION.

By C. S. Jackson.

1. WHEN a curve is to be determined, so far as may be, from the condition that $n+1$ ordinates at given equal intervals are to have given values, the slope and curvature at one of the given points, say P , may be required. (c.f. *Text-book of Gunnery*, 1902 ed., p. 174 *et seq.*) The object of this note is to express these quantities in a form which puts in direct evidence the effect of each given ordinate on the final result.

2. Numbering the given ordinates right and left from P , let h be the common interval, and let the parabolic curve of the n^{th} degree, passing through the given points, be $y=f(x)$.



Write $D(m)$ for $f(x+mh)-f(x)$. Then, by Taylor's theorem,

$$\left. \begin{aligned} D(-\lambda) &= -\lambda h f'(x) + \frac{\lambda^2 h^2}{2!} f''(x) \dots + \frac{(-\lambda h)^n}{n!} f^n(x) \\ \dots \dots \dots \\ D(-1) &= -h f'(x) + \frac{h^2}{2!} f''(x) \dots + \frac{(-h)^n}{n!} f^n(x) \\ D(1) &= h f'(x) + \frac{h^2}{2!} f''(x) \dots + \frac{h^n}{n!} f^n(x) \\ \dots \dots \dots \\ D(\mu) &= \mu h f'(x) + \frac{\mu^2 h^2}{2!} f''(x) \dots + \frac{\mu^n h^n}{n!} f^n(x) \end{aligned} \right\} \dots (1).$$

$$\lambda + \mu = n.$$

Solving these equations for $hf'(x)$ and $\frac{h^2}{2!}f''(x)$, we have

$$hf'(x) = \frac{\begin{vmatrix} D(-\lambda), & (-\lambda)^2, & (-\lambda)^3, & \dots, & (-\lambda)^n \\ \dots\dots\dots \\ D(\mu), & \mu^2, & \mu^3, & \dots, & \mu^n \end{vmatrix}}{\begin{vmatrix} -\lambda, & (-\lambda)^2, & (-\lambda)^3, & \dots, & (-\lambda)^n \\ \dots\dots\dots \\ \mu, & \mu^2, & \mu^3, & \dots, & \mu^n \end{vmatrix}} \dots\dots\dots (2),$$

$$\text{and } \frac{h^2}{2!}f''(x) = \frac{\begin{vmatrix} -\lambda, & D(-\lambda), & (-\lambda), & \dots, & (-\lambda)^n \\ \dots\dots\dots \\ \mu, & D(\mu), & \mu^3, & \dots, & \mu^n \end{vmatrix}}{\begin{vmatrix} -\lambda, & (-\lambda)^2, & (-\lambda)^3, & \dots, & (-\lambda)^n \\ \dots\dots\dots \\ \mu, & \mu^2, & \mu^3, & \dots, & \mu^n \end{vmatrix}} \dots\dots\dots (3).$$

3. The coefficients of $D(-\lambda)\dots$ in these formulæ may be evaluated thus.

Consider the determinants

$$A = \begin{vmatrix} x_1, & x_1^2, & x_1^n \\ \dots\dots\dots \\ x_n, & x_n^2, & x_n^n \end{vmatrix} = x_1, x_2, \dots, x_n \Pi (x_r - x_s),$$

$r = 1, 2, 3, \dots, n, \quad s = 1, 2, 3, \dots, n, \quad r > s,$

$$A_{p\kappa} = \begin{vmatrix} x_1, & x_1^{\kappa-1}, & x_1^{\kappa+1}, & \dots, & x_1^n \\ \dots\dots\dots \\ x_{p-1}, & \dots\dots\dots \\ x_{p+1}, & \dots\dots\dots \\ \dots\dots\dots \\ x_n, & \dots\dots\dots, & x_n^n \end{vmatrix} = \frac{x_1 \dots x_n}{x_p} \Pi (x_r - x_s) \times {}_pX_{n-\kappa},$$

$r = 1, 2, \dots, p-1, p+1, \dots, n, \quad s = 1, 2, \dots, p-1, p+1, \dots, n, \quad r > s,$
 where ${}_pX_{n-\kappa}$ is a symmetric function of the $n-1$ variables $x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n$ of degree $n-\kappa$, as is evident by consideration of dimensions.

Further $A_{p\kappa}$ is of degree n in x_m

$$\begin{array}{ccccccc} x_1 \dots x_n & , & , & 1 & , & , & \\ \Pi (x_r - x_s) & , & , & n-2 & , & , & \end{array}$$

and hence ${}_pX_{n-\kappa}$ is linear in x_m so that ${}_pX_{n-\kappa}$ is the sum of the combinations $n - \kappa$ together of the above $n - 1$ variables.

In the application of these results we put

$$\begin{aligned}x_1 &= -\lambda, \\x_2 &= -(\lambda - 1), \\&\dots\dots\dots, \\x_r &= -s, \\&\dots\dots\dots, \\x_\lambda &= -1, \\&\dots\dots\dots, \\x_p &= q, \\&\dots\dots\dots, \\x_n &= \mu,\end{aligned}$$

and we see that

$$\begin{aligned}A &= (-1)^\lambda \lambda! \mu! \frac{n!}{\mu} \frac{(n-1)!}{\mu-1} \dots \frac{\lambda!}{\lambda!} \cdot (\lambda-1)! \dots 1 \\&= (-1)^\lambda n! (n-1)! (n-2)! \dots = (-1)^\lambda n!! \text{ say } \dots\dots\dots (4),\end{aligned}$$

$$\begin{aligned}A_{r.1} &= (-1)^\lambda \frac{\lambda! \mu!}{-s} \cdot \frac{n!}{\mu \cdot \mu + s} \cdot \frac{(n-1)!}{(\mu-1)(\mu+s-1)} \dots \\&\dots \frac{(\lambda+1)!}{1(1+s)} \frac{\lambda!}{\lambda!} \cdot \frac{(\lambda-1)!}{s-1} \frac{(\lambda-s+1)!}{1} \frac{(\lambda-s)!}{(\lambda-s)!} (\lambda-s-1)! \dots 1 \cdot {}_rX_{n-1} \\&= (-1)^{\lambda-1} \frac{n!!}{(\lambda-s)! (\mu+s)!} {}_rX_{n-1} \\&= \frac{-A}{(\lambda-s)! (\mu+s)!} \frac{(-1)^\lambda \lambda! \mu!}{-s} \dots\dots\dots (5),\end{aligned}$$

$$\begin{aligned}A_{p.1} &= (-1)^\lambda \frac{\lambda! \mu!}{q} \cdot \frac{n!}{\mu (\mu-q)} \frac{(n-1)!}{(\mu-1)(\mu-q-1)} \dots \\&\dots \frac{(q+\lambda+1)!}{(q+1) \cdot 1} \frac{(q+\lambda)!}{(q+\lambda)!} \frac{(q+\lambda-1)!}{q-1} \dots \\&\dots \frac{(\lambda+1)!}{1} \frac{\lambda!}{\lambda!} \cdot (\lambda-1)! \dots 1 \cdot {}_pX_{n-1} \\&= \frac{(-1)^\lambda n!!}{(\mu-q)! (\lambda+q)!} {}_pX_{n-1} \\&= \frac{A}{(\mu-q)! (\lambda+q)!} \frac{(-1)^\lambda \lambda! \mu!}{q} \dots\dots\dots (6),\end{aligned}$$

and

$$\begin{aligned} A_{r.2} &= \frac{-A}{(\lambda-s)!(\mu+s)!} {}_rX_{n-2} = \frac{-A}{(\lambda-s)!(\mu+s)!} \frac{(-1)^\lambda \lambda! \mu!}{-s} \\ &\quad \times \left[\frac{1}{-\lambda} + \frac{1}{-(\lambda-1)} \dots + \frac{1}{-1} + \frac{1}{1} + \frac{1}{2} \dots + \frac{1}{\mu} + \frac{1}{s} \right] \\ &= A \cdot (-1)^\lambda \frac{\lambda! \mu!}{(\lambda-s)!(\mu+s)!} \cdot \frac{1}{s} \cdot \left[\sigma_\mu - \sigma_\lambda + \frac{1}{s} \right] \dots \dots (7), \end{aligned}$$

writing σ_μ for $1 + \frac{1}{2} \dots + \frac{1}{\mu}$,

and

$$\begin{aligned} A_{p.2} &= \frac{A}{(\lambda+q)!(\mu-q)!} {}_pX_{n-2} \\ &= A \cdot (-1)^\lambda \frac{\lambda! \mu!}{(\lambda+q)!(\mu-q)!} \frac{1}{q} \left[\sigma_\mu - \sigma_\lambda - \frac{1}{q} \right] \dots \dots (8). \end{aligned}$$

4. Expanding the determinants in (2) and (3), and evaluating the successive minors by putting $s=\lambda, \lambda-1, \dots, 1$ in turn in (5) and (7) and $q=1, 2, \dots, \mu$ in turn in (6) and (8) we find, paying due regard to sign,

$$\begin{aligned} hf'(x) &= \frac{\mu}{\lambda+1} \frac{D(1)}{1} - \frac{\mu \cdot (\mu-1)}{(\lambda+1)(\lambda+2)} \frac{D(2)}{2} \dots - \frac{\lambda! \mu!}{n!} (-1)^\mu \frac{D(\mu)}{\mu} \\ &\quad + \frac{\lambda}{\mu+1} \frac{D(-1)}{-1} - \frac{\lambda \cdot (\lambda-1)}{(\mu+1)(\mu+2)} \frac{D(-2)}{-2} \dots \\ &\quad - \frac{\lambda! \mu!}{n!} (-1)^\lambda \frac{D(-\lambda)}{-\lambda} \dots \dots \dots (9), \end{aligned}$$

and writing σ for $\sigma_\mu - \sigma_\lambda$

$$\begin{aligned} \frac{h^2}{2!} f''(x) &= - \frac{\mu}{\lambda+1} \frac{D(1)}{1} \left(\sigma - \frac{1}{1} \right) \\ &\quad + \frac{\mu(\mu-1)}{(\lambda+1)(\lambda+2)} \frac{D(2)}{2} \left(\sigma - \frac{1}{2} \right) \dots \\ &\quad + \frac{\lambda! \mu!}{n!} (-1)^\mu \frac{D\mu}{\mu} \left(\sigma - \frac{1}{\mu} \right) - \frac{\lambda}{\mu+1} \frac{D(-1)}{-1} \left(\sigma + \frac{1}{1} \right) \\ &\quad + \frac{\lambda \cdot (\lambda-1)}{(\mu+1)(\mu+2)} \cdot \frac{D(-2)}{-2} \left(\sigma + \frac{1}{2} \right) \dots \\ &\quad + \frac{\lambda! \mu!}{n!} (-1)^\lambda \frac{D(-\lambda)}{-\lambda} \left(\sigma + \frac{1}{\lambda} \right) \dots \dots \dots (10) \end{aligned}$$

$$\begin{aligned}
 = & -\sigma h f''(x) + \frac{\mu}{\lambda+1} \frac{D(1)}{1^2} - \frac{\mu \cdot (\mu-1)}{(\lambda+1)(\lambda+2)} \frac{D(2)}{2^2} \dots \\
 & - \frac{\lambda! \mu!}{n!} (-1)^\mu \frac{D(\mu)}{\mu^2} + \frac{\lambda}{\mu+1} \frac{D(-1)}{1^2} - \frac{\lambda \cdot \lambda-1}{\mu+1 \cdot \mu+2} \frac{D(-2)}{2^2} \dots \\
 & - \frac{\lambda! \mu!}{n!} (-1)^\lambda \frac{D(-\lambda)}{\lambda^2} \dots \dots \dots (11).
 \end{aligned}$$

If $\mu = n$ and so $\lambda = 0$ the formulæ at the initial point become

$$h f''(x) = n \frac{D(1)}{1} - \frac{n \cdot n-1}{1 \cdot 2} \frac{D(2)}{2} \dots - (-1)^n \frac{D(n)}{n} \dots (12),$$

$$\begin{aligned}
 \frac{h^2}{2!} f'''(x) = & -n \frac{D(1)}{1} \left(\sigma - \frac{1}{2}\right) + \frac{n \cdot n-1}{1 \cdot 2} \frac{D(2)}{2} \left(\sigma - \frac{1}{2}\right) \dots \\
 & + (-1)^n \frac{D(n)}{n} \left(\sigma - \frac{1}{n}\right) \dots \dots \dots (13)
 \end{aligned}$$

$$\begin{aligned}
 = & -\sigma h f''(x) + n \frac{D(1)}{1^2} - \frac{n \cdot n-1}{1 \cdot 2} \frac{D(2)}{n^2} \dots \\
 & - (-1)^n \frac{D(n)}{n^2} \dots \dots \dots (14).
 \end{aligned}$$

5. The precise object of these formulæ is, as has been already stated, to present the contribution of each separate piece of evidence separately. The formulæ thus express numerically the consequences in any particular case of the decision adopted of a question which frequently confronts the statistician.

We reach the most reliable conclusion by using all the relevant evidence. But what evidence is relevant?

If we know that all the experimental material is homogeneous, so to speak, then all the experimental results are relevant evidence towards finding the slope and curvature at any one experimental point. But if the material is continually varying the curve through the series of points is in the nature of an envelope and the true curve, in a given part of the range, connecting the variables, is better determined by the immediately adjacent points only than by including more remote points. In fact the more remote data are irrelevant to the question in hand.

6. The formulæ also indicate that the ordinary interpolation formulæ, to which they are equivalent, do in effect assign weights to the different ordinates.

Thus taking the numerical case $\lambda=2$, $\mu=3$, we have, by (9),

$$hf'(x) = D(1) - \frac{1}{4}\{D(2)\} + \frac{1}{30}\{D(3)\} - \frac{1}{2}\{D(-1)\} + \frac{1}{20}\{D(-2)\}$$

$$= y_1 - \frac{1}{4}y_2 + \frac{1}{30}y_3 - \frac{1}{2}y_{-1} + \frac{1}{20}y_{-2} - \frac{1}{3}y_0,$$

so that we are really attributing weights proportional to these coefficients to the several observed quantities, a fact to which attention is not explicitly called by the usual process of interpolation.

Professor Alfred Lodge has pointed out to me that the formulæ also illustrate the superior accuracy of interpolation by central differences over interpolation by exclusively advancing or receding differences.

Taking seven ordinates a, b, c, f, p, q, r , so that $n=6$, we find

$$60h \frac{da}{dx} = -147a + 360b - 450c + 400f - 225p + 72q - 10r,$$

$$60h \frac{df}{dx} = -a + 9b - 45c + 45p - 9q + r.$$

The large numerical multiples in the first case show how an error in one ordinate will produce a large effect on the result sought.

7. One or two simple illustrations may be added.

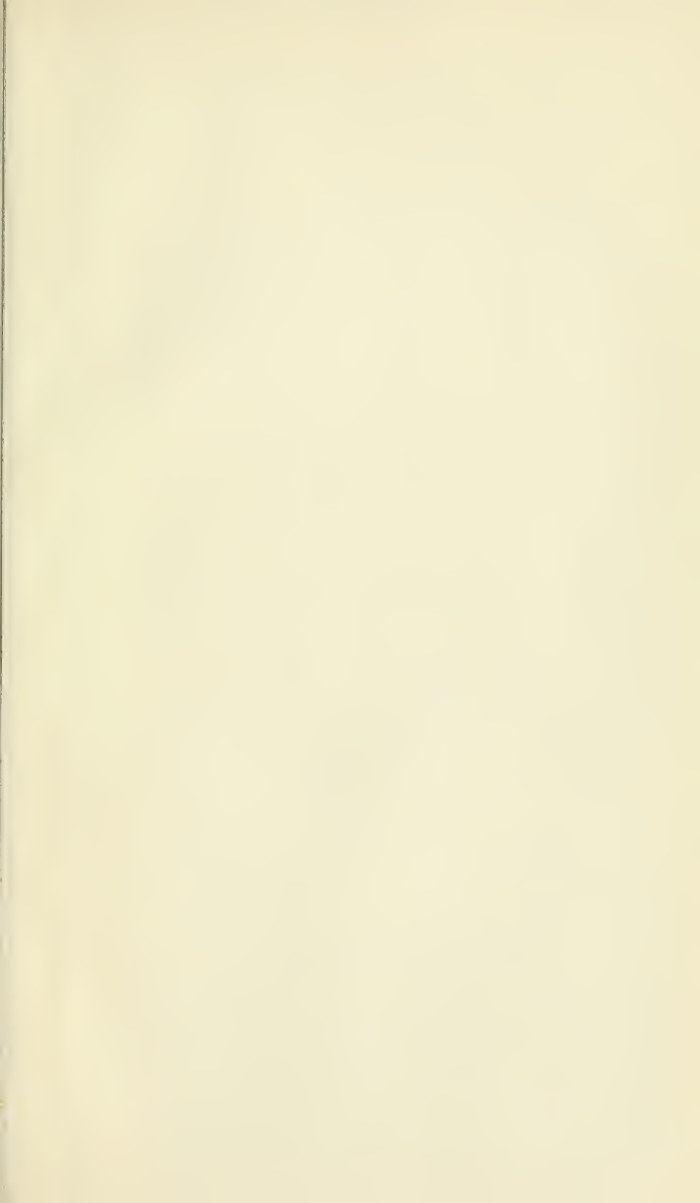
(1) From (12) it appears that an increase in any of the even ordinates will decrease $f'(x)$.

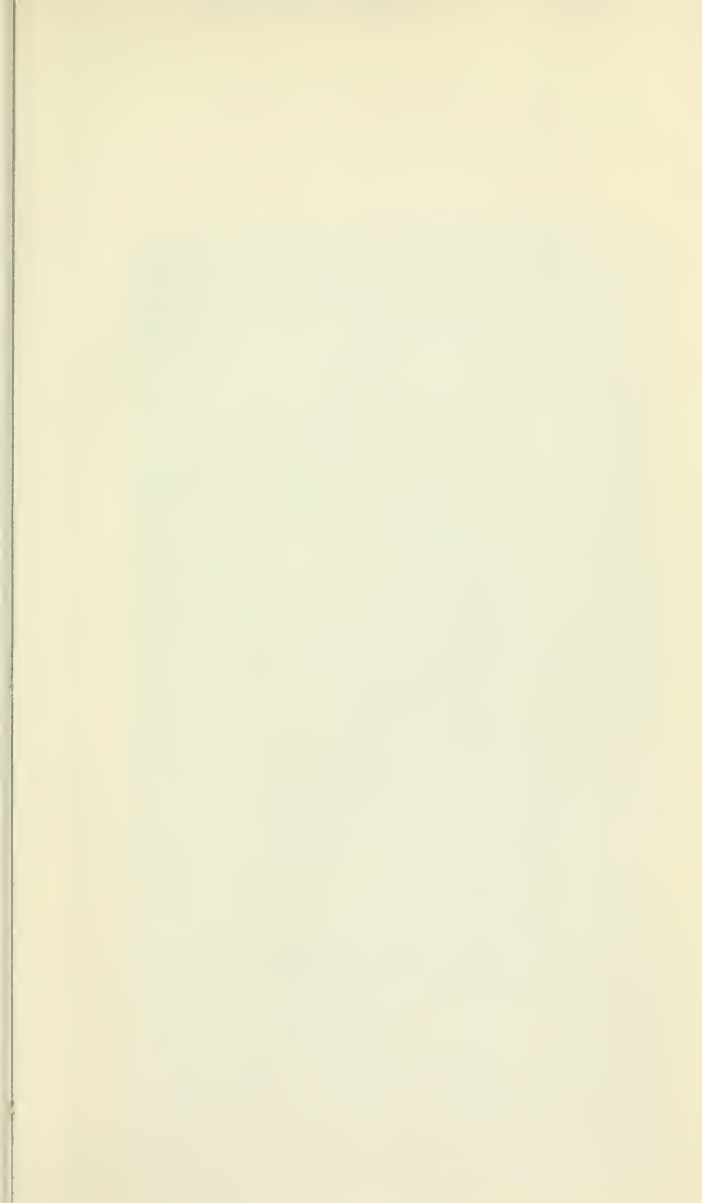
(2) We should expect a better forecast of this year's output of bicycles from the statistics of the last 3 or 4 years than from the statistics of the last 30 or 40 years.

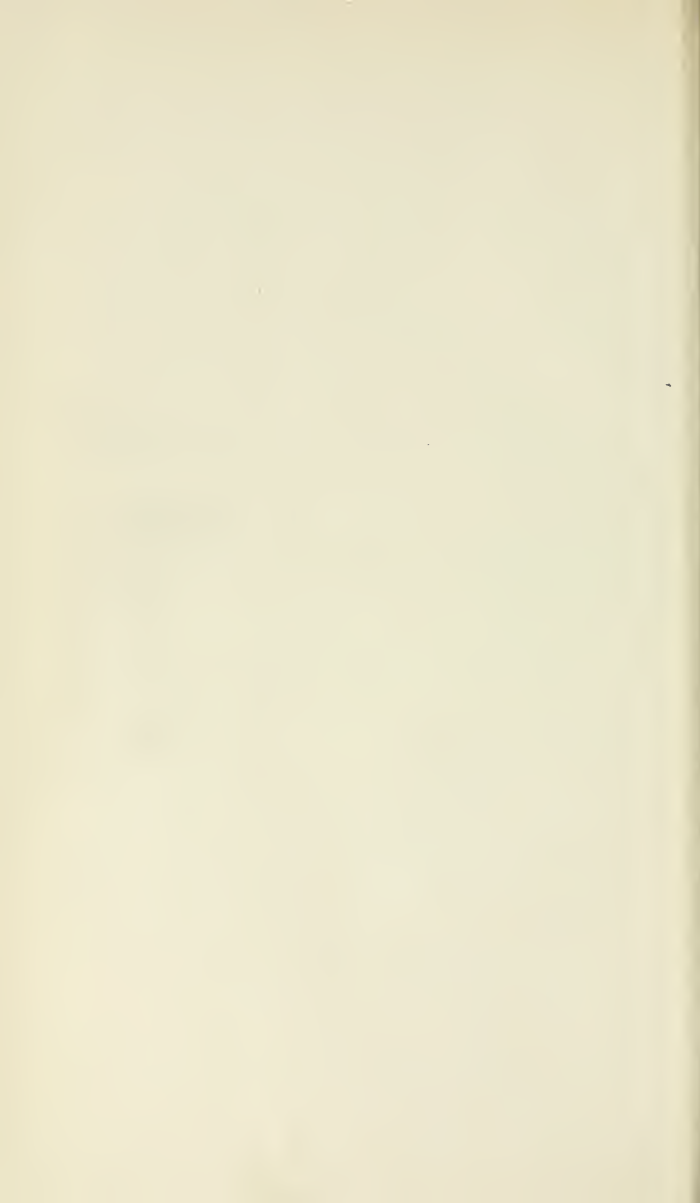
If we are in doubt whether to take 3 years, or 4 years, the formulæ will not decide the question: for it is a question of fact and not of mathematics; but they will disclose the consequences and the importance of the decision we adopt.

It is needless to mention that the ordinates are taken to be the direct results of experiment or observation, not smoothed by any process.

END OF VOL. XXXVIII.







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